ANALYSIS OF LAMINATED COMPOSITE STRUCTURES

Theory and Numerics

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Draft Edition 1

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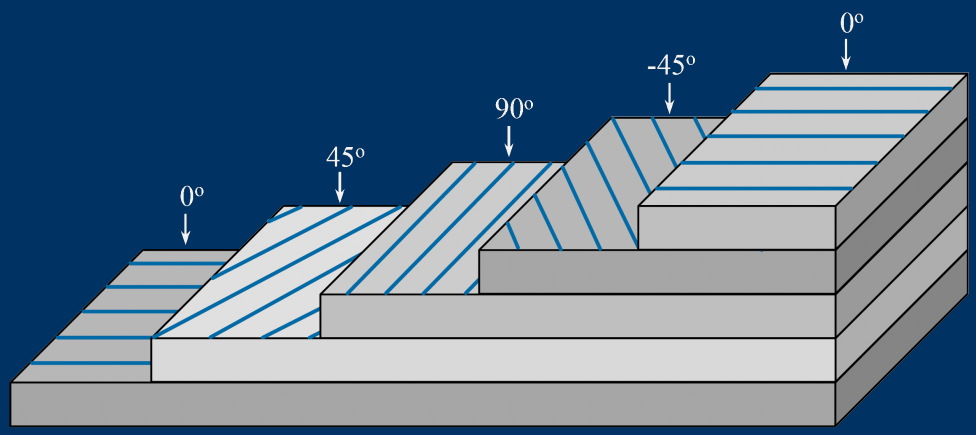

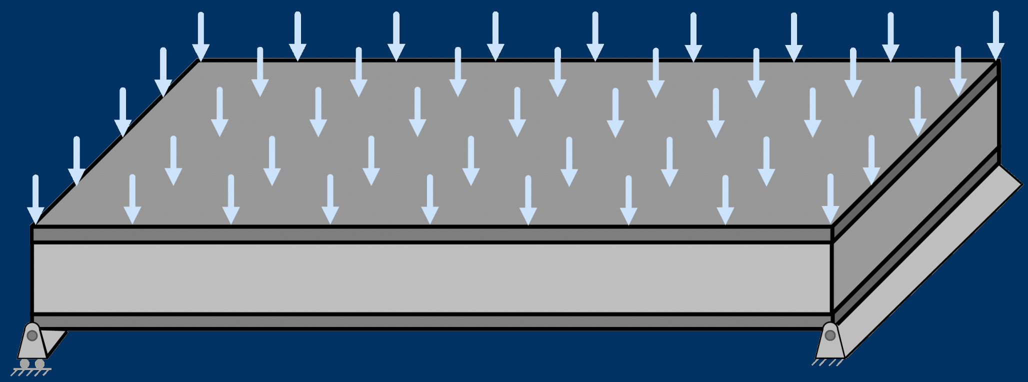

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This is a working draft of the first edition of the textbook that is currently being revised and updated.

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Part I: Fundamentals

Linearized Anisotropic Elasticity 1

1.1 Surface and internal forces

1.1.1 Traction vector

Let us consider a solid body that is subjected to external loads as shown in Fig. 1.1(a). In order to characterize the intensity of internal forces at a point $\boldsymbol{x}$, we section the body into two regions as shown in Fig. 1.1(b). Let us consider a small area Δa on the section with unit outward normal $\boldsymbol{n}$. The resultant

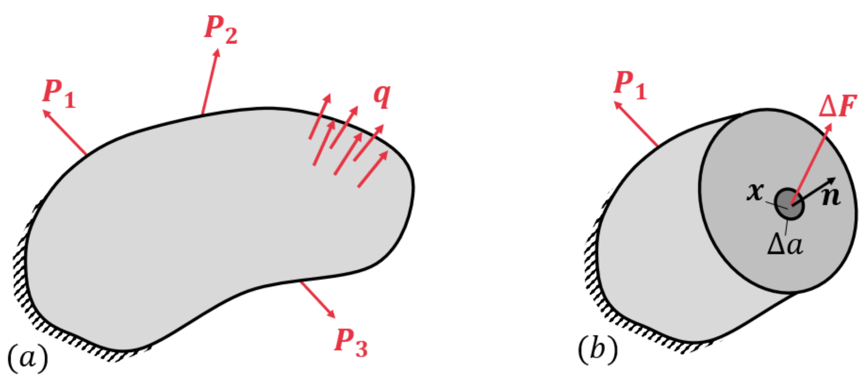


Figure 1.1: Forces acting on (a) a solid body and (b) a sectioned region of a solid body

force on the area Δa is denoted by $\Delta \boldsymbol{F}$. The traction vector $\boldsymbol{f}$ is defined as the resultant force per unit area.

$$\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{n}) = \lim_{\Delta a \to 0} \frac{\Delta \boldsymbol{F}}{\Delta a} \tag{1.1}$$

The traction vector characterizes the intensity of the internal force acting on a surface. Note that, in general, the traction vector depends on both the spatial location $\boldsymbol{x}$ and the unit normal $\boldsymbol{n}$.

The traction vector is a force per unit area although it is usually represented by a single arrow for the sake of convenience as shown in Fig. 1.2. In general, the traction vector $\boldsymbol{f}$ need not be parallel to the unit normal $\boldsymbol{n}$. It can be resolved into a component that is normal to the surface (*normal traction*) and a component that is parallel to the surface (*shear traction*). It can be systematically shown that the traction vector is an odd function of the normal vector. That is,

$$\boldsymbol{f}(\boldsymbol{x}, -\boldsymbol{n}) = -\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{n}) \tag{1.2}$$

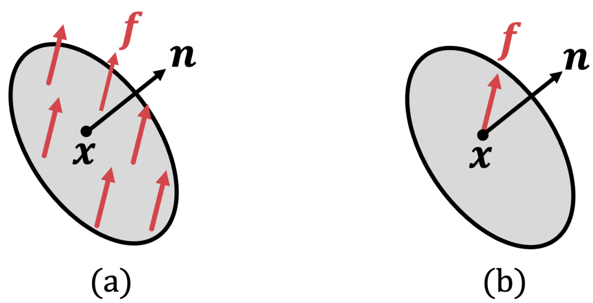


Figure 1.2: Representation of the traction vector as (a) a distributed load and (b) a single arrow

This is known as *Cauchy's Fundamental Lemma* and is equivalent to Newton's third law when applied to the opposite surfaces of a sectioned body.

1.1.2 Stress tensor

Consider a surface with normal $\boldsymbol{n}$ passing through point $\boldsymbol{x}$ in an elastic body. The *Cauchy stress theorem* states that there exists a second order tensor $\boldsymbol{\sigma}(\boldsymbol{x})$ called the Cauchy stress tensor that that relates the normal vector $\boldsymbol{n}$ to the traction vector $\boldsymbol{f}$ acting on the surface.

$$\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{n}) = \boldsymbol{\sigma}^T(\boldsymbol{x})\, \boldsymbol{n} \tag{1.3}$$

This can be written in matrix notation as follows,

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{21} & \sigma_{31} \\ \sigma_{12} & \sigma_{22} & \sigma_{32} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \tag{1.4}$$

where σ_{ij} are the components of the Cauchy stress tensor. Consider a volume element dv located at point $\boldsymbol{x}$ in a body as shown in Fig. 1.3(a). The components of the traction vector $\mathbf{f}$ acting on the surfaces of the element are obtained by substituting the components of the normal vector for each surface into Eqn. 1.4. For example, the traction vector acting on surface with normal oriented parallel to the positive x_1-direction, i.e., $\{\boldsymbol{n}\} = [1\ 0\ 0]^T$ is $\{\boldsymbol{t}\} = [\sigma_{11}\ \sigma_{12}\ \sigma_{13}]^T$. It can be shown using the balance of angular momentum at a point that the Cauchy stress tensor is symmetric, i.e., $\sigma_{ij} = \sigma_{ji}$.

Equation (1.4) can be expressed in compact matrix notation as follows

$$\{\boldsymbol{f}\} = [\boldsymbol{\sigma}]\{\boldsymbol{n}\} \tag{1.5}$$

The Cauchy stress tensor $\boldsymbol{\sigma}$ has units of N/m^2 (i.e. Pa) or lbf/in^2 (i.e. psi).

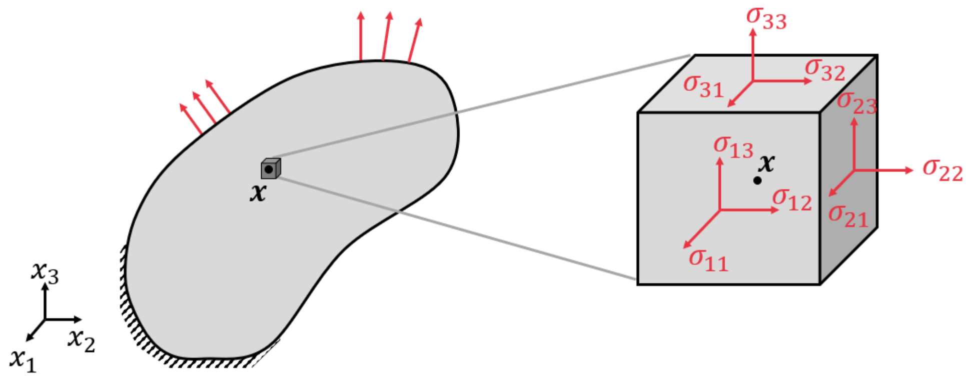


Figure 1.3: (a) Element in an body that is subjected to loads and (b) stress components acting on a volume element

1.2 Strain

Consider a material particle $\boldsymbol{P}$ that is initially located at $\boldsymbol{x}$ in the undeformed or reference configuration as depicted in Fig. 1.4.

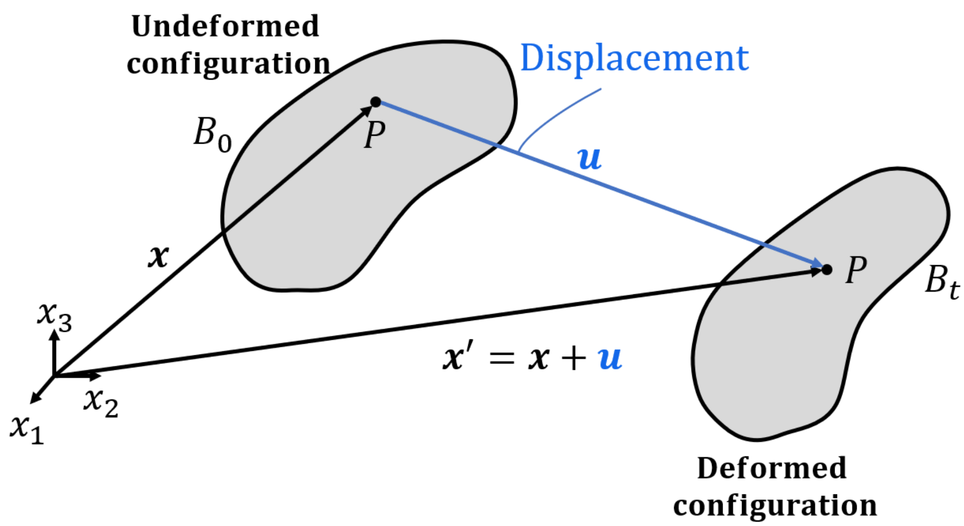


Figure 1.4: Deformation of an elastic body

The position vector of particle $\boldsymbol{P}$ in the deformed configuration is denoted by $\boldsymbol{x}'$. The displacement vector $\boldsymbol{u}$ of particle $\boldsymbol{P}$ is given by

$$\boldsymbol{u} = \boldsymbol{x}' - \boldsymbol{x} \tag{1.6}$$

In general, the displacement vector $\boldsymbol{u}$ will vary from point to point, and it also depends on time t, i.e.,

$$\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t) \tag{1.7}$$

In the case of quasi-static deformation, the position of a particle in the deformed configuration does not depend on time t and the displacement field is a function of the reference configuration coordinates

$$\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}) \tag{1.8}$$

where $\boldsymbol{u}(\boldsymbol{x})$ is the displacement field.

The strain tensor characterizes the intensity of deformation at a point. In the case of small deformations, the infinitesimal strain tensor $\boldsymbol{\varepsilon}$ is evaluated as follows [1].

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(\nabla\boldsymbol{u} + \nabla^T\boldsymbol{u}\right) \tag{1.9}$$

where $\nabla\boldsymbol{u}$ is the gradient of the displacement field with respect to the spatial coordinates $\mathbf{x}$ and the superscript T denotes its transpose.

The infinitesimal strain tensor is a symmetric second-order tensor. In the case of a Cartesian coordinate system, the components of the strain tensor are

$$\boldsymbol{\varepsilon_{ij}} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{1.10}$$

The six components of the strain tensor in a Cartesian coordinate system are ε_{11}, ε_{22}, ε_{33}, ε_{12}, ε_{23} and ε_{13}. The following are the three normal strains in the coordinate directions,

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1}, \quad \varepsilon_{22} = \frac{\partial u_2}{\partial x_2}, \quad \varepsilon_{33} = \frac{\partial u_3}{\partial x_3} \tag{1.11}$$

The strain components ε_{12}, ε_{23} and ε_{13} are the tensorial shear strains in each of the three coordinate planes

$$\begin{aligned}
\varepsilon_{12} &= \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) \\
\varepsilon_{23} &= \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \\
\varepsilon_{13} &= \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right)
\end{aligned} \tag{1.12}$$

The engineering shear strains γ_{ij} are twice the tensorial shear strains ε_{ij} and are given by

$$\gamma_{ij} = 2\varepsilon_{ij} \qquad (i \neq j) \tag{1.13}$$

The engineering shear strain γ_{ij} represents the reduction in angle between two material line elements that were originally parallel to the x_i and x_j-axes in the undeformed configuration. The shear strain is measured in radians.

1.3 Coordinate transformation

The components of the displacement vector $\boldsymbol{u}$, stress tensor $\boldsymbol{\sigma}$ and the strain tensor $\boldsymbol{\varepsilon}$ depend on the coordinate system used for the analysis. If the components of a vector or tensor are known in one coordinate system, the components in another coordinate system can be obtained through appropriate vector and tensor transformation rules. Consider two orthonormal coordinate systems (x_1, x_2, x_3) and (x_1', x_2', x_3') as shown in Fig. 1.5.

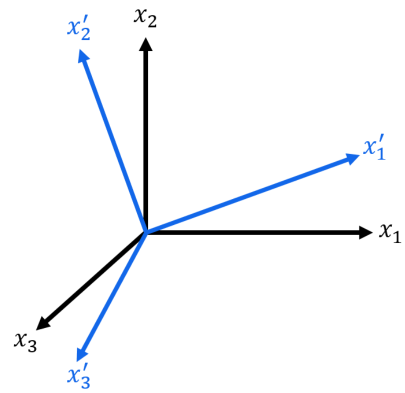


Figure 1.5: The primed and unprimed orthonormal coordinate systems

The two coordinate are related through the direction cosines matrix $\boldsymbol{A}$, the elements of which are defined as follows

$$A_{ij} = \cos(x_i', x_j) \tag{1.14}$$

where $\cos(x_i', x_j)$ represents the cosine of the angle between the coordinate axes x_i' and x_j. For example, $A_{12} = \cos(x_1', x_2)$ is the cosine of the angle between x_1' and x_2.

EXAMPLE 1.1: Direction cosine matrix

Consider two coordinate systems in which the x_3' and x_3 axes aligned in the same direction and the x_1' is oriented at an angle θ relative to the x_1 axis as shown in Fig. 1.6.

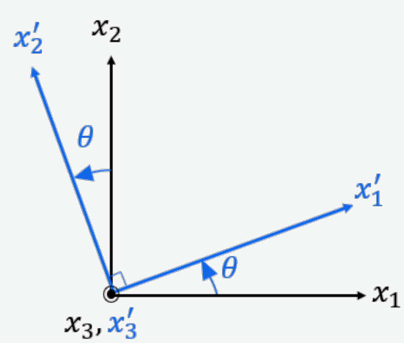


Figure 1.6: Orientation of the primed coordinate system relative to the unprimed coordinate system for 2D tensor transformations

In this case, the direction cosine matrix,

$$\begin{aligned}[A] &= \begin{bmatrix} \cos(x_1', x_1) & \cos(x_1', x_2) & \cos(x_1', x_3) \\ \cos(x_2', x_1) & \cos(x_2', x_2) & \cos(x_2', x_3) \\ \cos(x_3', x_1) & \cos(x_3', x_2) & \cos(x_3', x_3) \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta) & \cos(90^\circ - \theta) & \cos(90^\circ) \\ \cos(90^\circ + \theta) & \cos(\theta) & \cos(90^\circ) \\ \cos(90^\circ) & \cos(90^\circ) & \cos(0^\circ) \end{bmatrix}\end{aligned} \tag{1.15}$$

which can be written as

$$[A] = \begin{bmatrix} m & n & 0 \\ -n & m & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.16}$$

where $m = \cos\theta$ and $n = \sin\theta$.

1.3.1 Transformation of vectors

A vector, such as the displacement, can be written in terms of the components in either the primed or the unprimed coordinate systems

$$\boldsymbol{u} = u_1\boldsymbol{e}_1 + u_2\boldsymbol{e}_2 + u_3\boldsymbol{e}_3 = u_1'\boldsymbol{e}_1' + u_2'\boldsymbol{e}_2' + u_3'\boldsymbol{e}_3' \tag{1.17}$$

where u_i and u_i' are the components of the vector $\boldsymbol{u}$ in the unprimed and primed coordinate system, respectively. The components of a vector in the primed coordinate system can be obtained from the components in the unprimed coordinate system using the direction cosines matrix as follows

$$\{u'\} = [A]\{u\} \tag{1.18}$$

where $\{u'\}$ and $\{u\}$ are 3×1 column arrays of components in the primed and unprimed coordinate system, respectively.

1.3.2 Transformation of second-order tensors

The components of a second-order tensor $\boldsymbol{T}$ transform as follows between two different coordinate systems

$$[T'] = [A][T][A]^T \tag{1.19}$$

where $[T]$ and $[T']$ are the components of the second-order tensor $\boldsymbol{T}$ in the unprimed and primed coordinate systems, respectively. Equation (1.19) can be expressed in component form as follows

$$T'_{ij} = \sum_{p=1}^{3}\sum_{q=1}^{3} A_{ip}A_{jq}T_{pq} \tag{1.20}$$

1.4 Voigt notation

1.4.1 Stresses

In order to make it easier to perform the stress and strain transformation, we define the Voigt contracted notation for the stress components

$$\begin{aligned} &\sigma_1 \equiv \sigma_{11}, \quad \sigma_2 \equiv \sigma_{22}, \quad \sigma_3 \equiv \sigma_{33} \\ &\sigma_4 \equiv \sigma_{23}, \quad \sigma_5 \equiv \sigma_{13}, \quad \sigma_6 \equiv \sigma_{12} \end{aligned} \tag{1.21}$$

The contracted stress components can be arranged in the form of a column array as follows,

$$\{\boldsymbol{\sigma}\} = \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} \tag{1.22}$$

1.4.2 Strains

The strain components are contracted in the following manner,

$$\begin{aligned} &\varepsilon_1 \equiv \varepsilon_{11}, \quad \varepsilon_2 \equiv \varepsilon_{22}, \quad \varepsilon_3 \equiv \varepsilon_{33} \\ &\varepsilon_4 \equiv \gamma_{23} \equiv 2\varepsilon_{23}, \quad \varepsilon_5 \equiv \gamma_{13} \equiv 2\varepsilon_{13}, \quad \varepsilon_6 \equiv \gamma_{12} \equiv 2\varepsilon_{12} \end{aligned} \tag{1.23}$$

where γ_{ij} are the engineering shear strains.

The contracted strain components can be arranged in the form of a column array as follows,

$$\{\boldsymbol{\sigma}\} = \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} \tag{1.24}$$

1.5 Transformation of stresses and strains

1.5.1 Transformation of stress components

The components of the stress tensor transform as follows

$$[\sigma'] = [A][\sigma][A]^T \tag{1.25}$$

where $[\sigma]$ and $[\sigma']$ are the components of the stress tensor $\boldsymbol{\sigma}$ in the unprimed and primed coordinate systems, respectively. Equation (1.25) can be expressed in component form as follows

$$\sigma'_{ij} = \sum_{p=1}^{3}\sum_{q=1}^{3} A_{ip}A_{jq}\sigma_{pq} \tag{1.26}$$

For example,

$$\begin{aligned} \sigma'_{11} &= \sum_{p=1}^{3}\sum_{q=1}^{3} A_{1p}A_{1q}\sigma_{pq} \\ &= A_{11}^2\sigma_{11} + A_{11}A_{12}\sigma_{12} + A_{11}A_{13}\sigma_{13} \\ &+ A_{12}A_{11}\sigma_{21} + A_{12}^2\sigma_{22} + A_{12}A_{13}\sigma_{23} \\ &+ A_{13}A_{11}\sigma_{31} + A_{13}A_{12}\sigma_{32} + A_{13}^2\sigma_{33} \end{aligned} \tag{1.27}$$

Using the Voigt contracted notation (1.21), the stress transformation relationship for σ'_{11} in Eqn.1.27 can be expressed in terms of contracted stresses

$$\sigma'_1 = A_{11}^2\sigma_1 + A_{12}^2\sigma_2 + A_{13}^2\sigma_3 + 2A_{12}A_{13}\sigma_4 + 2A_{11}A_{13}\sigma_5 + 2A_{11}A_{12}\sigma_6 \tag{1.28}$$

Using the Voigt contracted notation, the stress transformation Eqn. (1.26) can be written as

$$\{\sigma'\} = [T_\sigma]\{\sigma\} \tag{1.29}$$

where $[T_\sigma]$ is the 6×6 stress transformation matrix

$$[T_\sigma] = \begin{bmatrix} A_{11}^2 & A_{12}^2 & A_{13}^2 & 2A_{12}A_{13} & 2A_{13}A_{11} & 2A_{11}A_{12} \\ A_{21}^2 & A_{22}^2 & A_{23}^2 & 2A_{22}A_{23} & 2A_{23}A_{21} & 2A_{21}A_{22} \\ A_{31}^2 & A_{32}^2 & A_{33}^2 & 2A_{32}A_{33} & 2A_{33}A_{31} & 2A_{31}A_{32} \\ A_{21}A_{31} & A_{22}A_{32} & A_{23}A_{33} & A_{22}A_{33}+A_{23}A_{32} & A_{21}A_{33}+A_{23}A_{31} & A_{22}A_{31}+A_{21}A_{32} \\ A_{31}A_{11} & A_{32}A_{12} & A_{33}A_{13} & A_{12}A_{33}+A_{13}A_{32} & A_{13}A_{31}+A_{11}A_{33} & A_{11}A_{32}+A_{12}A_{31} \\ A_{11}A_{21} & A_{12}A_{22} & A_{13}A_{23} & A_{12}A_{23}+A_{13}A_{22} & A_{13}A_{21}+A_{11}A_{23} & A_{11}A_{22}+A_{12}A_{21} \end{bmatrix} \tag{1.30}$$

whose A_{ij} are the components of the direction cosines matrix.

1.5.2 Transformation of strain components

The components of the strain tensor transform as follows

$$[\varepsilon'] = [A][\varepsilon][A]^T \tag{1.31}$$

where $[\varepsilon]$ and $[\varepsilon']$ are the components of the strain tensor $\boldsymbol{\varepsilon}$ in the unprimed and primed coordinate systems, respectively. The ε_1 component of the strain tensor transforms as follows

$$\varepsilon_1' = A_{11}^2\varepsilon_1 + A_{12}^2\varepsilon_2 + A_{13}^2\varepsilon_3 + A_{12}A_{13}\varepsilon_4 + A_{11}A_{13}\varepsilon_5 + A_{11}A_{12}\varepsilon_6 \tag{1.32}$$

This can be written as

$$\{\varepsilon'\} = [T_\varepsilon]\{\varepsilon\} \tag{1.33}$$

where $[T_\varepsilon]$ is the 6×6 strain transformation matrix

$$[T_\varepsilon] = \begin{bmatrix} A_{11}^2 & A_{12}^2 & A_{13}^2 & A_{12}A_{13} & A_{13}A_{11} & A_{11}A_{12} \\ A_{21}^2 & A_{22}^2 & A_{23}^2 & A_{22}A_{23} & A_{23}A_{21} & A_{21}A_{22} \\ A_{31}^2 & A_{32}^2 & A_{33}^2 & A_{32}A_{33} & A_{33}A_{31} & A_{31}A_{32} \\ 2A_{21}A_{31} & 2A_{22}A_{32} & 2A_{23}A_{33} & A_{22}A_{33}+A_{23}A_{32} & A_{21}A_{33}+A_{23}A_{31} & A_{22}A_{31}+A_{21}A_{32} \\ 2A_{31}A_{11} & 2A_{32}A_{12} & 2A_{33}A_{13} & A_{12}A_{33}+A_{13}A_{32} & A_{13}A_{31}+A_{11}A_{33} & A_{11}A_{32}+A_{12}A_{31} \\ 2A_{11}A_{21} & 2A_{12}A_{22} & 2A_{13}A_{23} & A_{12}A_{23}+A_{13}A_{22} & A_{13}A_{21}+A_{11}A_{23} & A_{11}A_{22}+A_{12}A_{21} \end{bmatrix} \tag{1.34}$$

The strain transformation matrix $[T_\varepsilon]$ is related to the stress transformation matrix $[T_\sigma]$ as follows

$$[T_\varepsilon] = [T_\sigma]^{-T} \tag{1.35}$$

EXAMPLE 1.2: 2D Coordinate transformation

In the case of a 2D coordinate transformation corresponding to a rotation of the coordinate system about the x_3-axis by an angle θ shown in Fig. 1.6 the direction cosines matrix is (refer Eqn. (1.16))

$$[A] = \begin{bmatrix} m & n & 0 \\ -n & m & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.36}$$

where $m = \cos\theta$ and $n = \sin\theta$. The stress transformation matrix for 2D coordinate transformation is obtained by substituting the components of the direction cosines matrix (1.36) into the stress transformation matrix (1.30)

$$[T_\sigma] = \begin{bmatrix} m^2 & n^2 & 0 & 0 & 0 & 2mn \\ n^2 & m^2 & 0 & 0 & 0 & -2mn \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & m & -n & 0 \\ 0 & 0 & 0 & n & m & 0 \\ -mn & mn & 0 & 0 & 0 & (m^2 - n^2) \end{bmatrix} \tag{1.37}$$

The strain transformation matrix for 2D coordinate transformations, obtained from (1.34) and (1.36) is

$$[T_\varepsilon] = \begin{bmatrix} m^2 & n^2 & 0 & 0 & 0 & mn \\ n^2 & m^2 & 0 & 0 & 0 & -mn \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & m & -n & 0 \\ 0 & 0 & 0 & n & m & 0 \\ -2mn & 2mn & 0 & 0 & 0 & (m^2 - n^2) \end{bmatrix} \tag{1.38}$$

1.6 Constitutive equations

1.6.1 Elastic stiffnesses

The generalized Hooke's law for an anisotropic material is expressed as

$$\sigma_{ij} = \sum_{k=1}^{3} \sum_{l=1}^{3} C_{ijkl} \varepsilon_{kl} \tag{1.39}$$

where C_{ijkl} are elastic constants which depend on the material. C_{ijkl} are the components of a fourth-order tensor known as the elastic stiffness tensor. Since the stress and strain tensors are symmetric, the stiffness tensor exhibits the following symmetries,

$$C_{ijkl} = C_{jikl} = C_{ijlk} \tag{1.40}$$

In contracted notation, Eqn. (1.39) can be written as

$$\sigma_p = \sum_{q=1}^{6} C_{pq} \varepsilon_q \tag{1.41}$$

where C_{pq} are the elastic stiffnesses in contracted notation with

$$C_{pq} = C_{\underbrace{ij}\underbrace{kl}} \tag{1.42}$$

and indices p and q are the Voigt contractions of ij and kl, respectively,

ij or kl	$\mapsto$	p or q
11		1
22		2
33		3
23 or 32		4
13 or 31		5
12 or 21		6

The stress-strain relationships (1.41) can be expressed in matrix form as

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} \tag{1.43}$$

Eqn. (1.43) can be written in compact form as

$$\{\boldsymbol{\sigma}\} = [C]\,\{\boldsymbol{\varepsilon}\} \tag{1.44}$$

where $[C]$ is a 6×6 matrix and is known as the *elastic stiffness matrix*.

1.6.2 Elastic Compliances

Hooke's law for an anisotropic material can be written in the alternate form,

$$\{\varepsilon\} = [S]\,\{\sigma\} \tag{1.45}$$

where $[S]$ is a 6×6 matrix and is known as the *elastic compliance matrix*. The compliance matrix is the inverse of the stiffness matrix, i.e.

$$[S] = [C]^{-1} \tag{1.46}$$

Eqn. (1.45) can be written in component form as

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} \tag{1.47}$$

1.6.3 Strain energy density

The strain energy density U is the elastic potential energy stored per unit volume due to deformation.

$$\begin{aligned} U &= \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}\sum_{i=1}^{6}\sigma_i\varepsilon_i \\ &= \frac{1}{2}\{\sigma\}^T\{\varepsilon\} = \frac{1}{2}\{\varepsilon\}^T\{\sigma\} \end{aligned} \tag{1.48}$$

Substitution of the stress σ from Hooke's law (1.44) into (1.48), gives

$$U = \frac{1}{2}\left([C]\{\varepsilon\}\right)^T\{\varepsilon\} = \frac{1}{2}\{\varepsilon\}^T\left([C]\{\varepsilon\}\right) \tag{1.49}$$

which implies that the following relationship must hold for all strains,

$$\begin{aligned} \frac{1}{2}\{\varepsilon\}^T[C]^T\{\varepsilon\} &= \frac{1}{2}\{\varepsilon\}^T[C]\{\varepsilon\} \\ \frac{1}{2}\{\varepsilon\}^T\left([C]^T - [C]\right)\{\varepsilon\} &= 0 \\ [C]^T &= [C] \end{aligned} \tag{1.50}$$

In other words, the 6×6 elastic stiffness matrix is symmetric, i.e., $C_{ij} = C_{ji}$.

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \tag{1.51}$$

In general, there are 21 independent elastic constants for an elastic material. Since the elastic stiffness matrix $[C]$ is symmetric, its inverse, the elastic compliance matrix $[S]$, is also symmetric and can be written as

$$[S] = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{12} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{13} & S_{23} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{14} & S_{24} & S_{34} & S_{44} & S_{45} & S_{46} \\ S_{15} & S_{25} & S_{35} & S_{45} & S_{55} & S_{56} \\ S_{16} & S_{26} & S_{36} & S_{46} & S_{56} & S_{66} \end{bmatrix} \tag{1.52}$$

1.7 Transformation of elastic stiffnesses and compliances

The constitutive equation (1.44) in the unprimed coordinate system is transformed to the primed coordinate system as follows,

$$\begin{aligned}\{\sigma\} &= [C]\{\,\varepsilon\} \\ [T_\sigma]^{-1}\{\sigma'\} &= [C][T_\varepsilon]^{-1}\{\varepsilon'\} \\ \{\sigma'\} &= [T_\sigma][C][T_\varepsilon]^{-1}\{\varepsilon'\} \\ &= [T_\sigma][C][T_\sigma]^T\{\varepsilon'\}\end{aligned} \tag{1.53}$$

which can be written as

$$\{\sigma'\} = [C']\{\varepsilon'\} \tag{1.54}$$

where $[C']$ is the elastic stiffness matrix in the primed coordinate system

$$[C'] = [T_\sigma]\,[C]\,[T_\sigma]^T \tag{1.55}$$

Similarly, it can be shown that the elastic compliance matrix transforms as follows

$$[S'] = [T_\varepsilon]\,[S]\,[T_\varepsilon]^T \tag{1.56}$$

1.8 Material symmetry

In the most general case, an anisotropic elastic material has 21 independent material constants. However, it may have fewer independent elastic constants if it exhibits symmetries.

1.8.1 Monoclinic materials

Monoclinic materials have one plane of reflectional symmetry. Consider a monoclinic material that exhibits reflectional symmetry about the $x_1 - x_2$ coordinate plane. Let's assume that it subjected to an axial strain ε_1 and a shear strain $\varepsilon_5(=\gamma_{13})$ as depicted in Fig. 1.7(a) with all other strain components being zero.

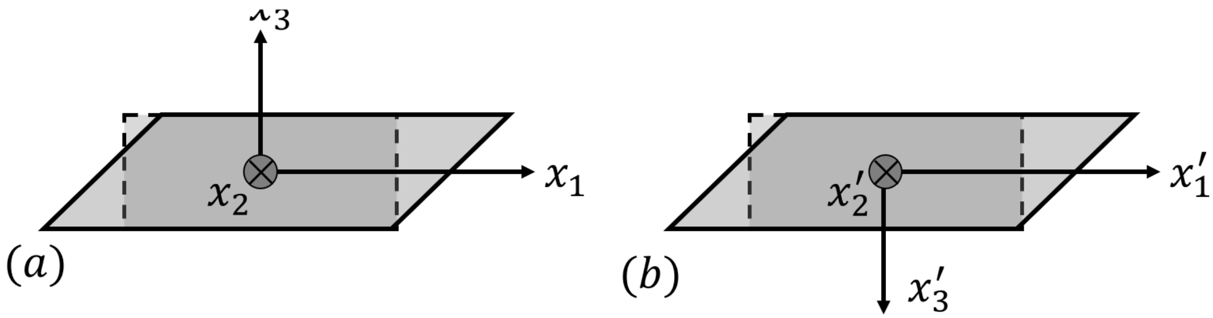


Figure 1.7: Monoclinic material subjected to normal strain in the x_1 direction and shear strain in the $x_1 - x_3$ plane (a) prior to reflection and (b) after reflection

The strain energy density (1.48) can be written in terms of strain components,

$$U = \frac{1}{2}\sum_{p=1}^{6}\sigma_p\varepsilon_p = \frac{1}{2}\sum_{p=1}^{6}\sum_{q=1}^{6}C_{pq}\varepsilon_p\varepsilon_q \tag{1.57}$$

The strain energy density corresponding to the deformation in Fig. 1.7(a) is

$$U = \frac{1}{2}C_{11}\varepsilon_1^2 + C_{15}\varepsilon_1\varepsilon_5 + \frac{1}{2}C_{55}\varepsilon_5^2 \tag{1.58}$$

Next, the material is reflected about its symmetry plane and subjected to the same deformation, as shown in Fig. 1.7(b). The strain energy density is unaffected since the material has been reflected about its symmetry plane prior to deformation. Thus,

$$U = \frac{1}{2}C'_{11}(\varepsilon'_1)^2 + C'_{15}\varepsilon'_1\varepsilon'_5 + \frac{1}{2}C'_{55}(\varepsilon'_5)^2 \tag{1.59}$$

The elastic stiffnesses remain the same when a monoclinic material is reflected about its symmetry plane. In addition, the normal strain in the reflected coordinate system remains the same while the shear strain changes sign, i.e., $\varepsilon'_1 = \varepsilon_1$ and $\varepsilon'_5 = -\varepsilon_5$. Therefore, the strain energy density

$$\begin{aligned} U &= \frac{1}{2}C_{11}\varepsilon_1^2 + C_{15}\varepsilon_1(-\varepsilon_5) + \frac{1}{2}C_{55}(-\varepsilon_5)^2 \\ &= \frac{1}{2}C_{11}\varepsilon_1^2 - C_{15}\varepsilon_1\varepsilon_5 + \frac{1}{2}C_{55}\varepsilon_5^2 \end{aligned} \tag{1.60}$$

Subtracting (1.60) from (1.58) gives

$$2C_{15}\varepsilon_1\varepsilon_5 = 0 \tag{1.61}$$

Since this result has to hold true for all ε_1 and ε_5, we infer that the elastic stiffness $C_{15} = 0$. Using a similar argument, it can be shown that $C_{14} = C_{24} = C_{25} = C_{34} = C_{35} = C_{46} = C_{56} = 0$ for monoclinic materials. Thus, the elastic stiffness matrix $[C]$ reduces to

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26} \\ C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{45} & C_{55} & 0 \\ C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66} \end{bmatrix} \tag{1.62}$$

A monoclinic material has 13 independent material constants.

1.8.2 Orthotropic materials

Orthotropic materials have three orthogonal planes of reflection symmetry and the elastic stiffness matrix has the following form

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \tag{1.63}$$

Orthotropic materials have 9 independent material constants.

1.8.3 Transversely isotropic materials

Transversely isostropic materials exhibit rotational symmetry about an axis as shown in Fig. (1.8).

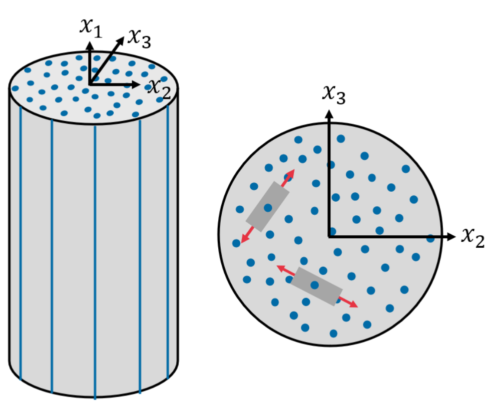


Figure 1.8: Transversely isotropic material

The elastic stiffness matrix has the following form if a material exhibits rotational symmetry about the x_1 axis,

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{12} & C_{23} & C_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{66} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \tag{1.64}$$

Note: $C_{44} = \frac{1}{2}(C_{22} - C_{23})$. Transversely isotropic materials have 5 independent material constants.

1.8.4 Isotropic materials

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{66} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{66} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \tag{1.65}$$

where $C_{66} = \frac{1}{2}(C_{11} - C_{12})$. Isotropic materials have 2 independent material constants.

1.9 Engineering constants

Consider an orthotropic material with reflectional symmetry about the coordinate planes.

(a) When an orthotropic material is subjected to a normal stress σ_1 with all other stresses being zero, the normal strain in the x_1 direction is

$$\varepsilon_1 = \frac{\sigma_1}{E_1} \tag{1.66}$$

where E_i is the Young's modulus in the x_i- direction. The Poisson's ratio is defined as

$$\nu_{ij} = -\frac{\varepsilon_j}{\varepsilon_i} \quad \text{when subjected to stress } \sigma_i \tag{1.67}$$

When subjected to a stress σ_1, the Poisson's ratio ν_{12} is defined as

$$\nu_{12} = -\frac{\varepsilon_2}{\varepsilon_1} \tag{1.68}$$

Thus the transverse normal strain ε_2 induced by the normal stress σ_1 is

$$\varepsilon_2 = -\nu_{12}\varepsilon_1 = -\nu_{12}\frac{\sigma_1}{E_1} = -\frac{\nu_{12}}{E_1}\sigma_1 \tag{1.69}$$

Similarly, the transverse normal strain ε_3 is

$$\varepsilon_3 = -\nu_{13}\frac{\sigma_1}{E_1} = -\frac{\nu_{13}}{E_1}\sigma_1 \tag{1.70}$$

When an orthotropic material is subjected to a normal stress σ_2 with all other stresses being zero, the normal strains induced are

$$\varepsilon_1 = -\frac{\nu_{21}}{E_2}\sigma_2, \quad \varepsilon_2 = \frac{\sigma_2}{E_2}, \quad \varepsilon_3 = -\frac{\nu_{23}}{E_2}\sigma_2 \tag{1.71}$$

Similarly, when an orthotropic material is subjected to a normal stress σ_3 with all other stresses being zero, the normal strains induced are

$$\varepsilon_1 = -\frac{\nu_{31}}{E_3}\sigma_3, \quad \varepsilon_2 = -\frac{\nu_{32}}{E_3}\sigma_3, \quad \varepsilon_3 = \frac{\sigma_3}{E_3} \tag{1.72}$$

When subjected to all three normal stresses, the normal strains are

$$\begin{aligned}
\varepsilon_1 &= \frac{1}{E_1}\sigma_1 - \frac{\nu_{21}}{E_2}\sigma_2 - \frac{\nu_{31}}{E_3}\sigma_3 \\
\varepsilon_2 &= -\frac{\nu_{12}}{E_1}\sigma_1 + \frac{1}{E_2}\sigma_2 - \frac{\nu_{32}}{E_3}\sigma_3 \\
\varepsilon_3 &= -\frac{\nu_{13}}{E_1}\sigma_1 - \frac{\nu_{23}}{E_2}\sigma_2 + \frac{1}{E_3}\sigma_3
\end{aligned} \tag{1.73}$$

(b) When an orthotropic material is subjected to a shear stress τ_{23}, the shear strain

$$\gamma_{23} = \frac{\tau_{23}}{G_{23}} \tag{1.74}$$

where G_{23} is the shear modulus in the $x_2 - x_3$ plane. This relationship can be written in contracted notation as follows

$$\varepsilon_4 = \frac{1}{G_{23}}\sigma_4 \tag{1.75}$$

Similarly, the other shear strains are

$$\begin{aligned}
\varepsilon_5 &= \frac{1}{G_{13}}\sigma_5 \\
\varepsilon_6 &= \frac{1}{G_{12}}\sigma_6
\end{aligned} \tag{1.76}$$

where G_{13} and G_{12} are the shear moduli in the $x_1 - x_3$ and $x_1 - x_2$ planes, respectively. Equations (1.73),(1.75), (1.76) can be combined into a single matrix equation as follows

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} \tag{1.77}$$

The 6×6 matrix in (1.77) is the compliance matrix $[S]$ in terms of the nine engineering constants. Since the compliance matrix is symmetric,

$$\frac{\nu_{12}}{E_1} = \frac{\nu_{21}}{E_2} \tag{1.78}$$

Similar relationships exist that relate the other Poisson's ratios and the Young's moduli,

$$\frac{\nu_{ij}}{E_i} = \frac{\nu_{ji}}{E_j}, \qquad i \neq j \tag{1.79}$$

Equations (1.79) are known as the *reciprocity relations*. Note that $\nu_{12} \neq \nu_{21}$ if $E_1 \neq E_2$. In general, ν_{21} and ν_{12} are different but ν_{21} can be calculated from ν_{12} and the Young's moduli E_1 and E_2.

An orthotropic material has a total of 9 engineering constants, namely E_1, E_2, E_3, ν_{12}, ν_{13}, ν_{23}, G_{12}, G_{13} and G_{23}. The elastic compliances for an orthotropic material can be expressed in terms of the engineering constants as

$$\begin{gathered} S_{11} = \frac{1}{E_1}, \quad S_{12} = -\frac{\nu_{12}}{E_1}, \quad S_{13} = -\frac{\nu_{13}}{E_1}, \\ S_{22} = \frac{1}{E_2}, \quad S_{23} = -\frac{\nu_{23}}{E_2}, \quad S_{33} = \frac{1}{E_3}, \\ S_{44} = \frac{1}{G_{23}}, \quad S_{55} = \frac{1}{G_{13}}, \quad S_{66} = \frac{1}{G_{12}} \end{gathered} \tag{1.80}$$

The stiffness matrix $[C]$ can be obtained by inverting the 6×6 compliance matrix $[S]$.

$$[C] = [S]^{-1} \tag{1.81}$$

A transversely isotropic material is a special case of an orthotropic material. The engineering constants in the case of a transversely isotropic material with rotational symmetry about the x_1 axis are

$$\begin{gathered} E_1, \quad E_2 = E_3, \quad \nu_{12} = \nu_{13}, \quad \nu_{23}, \\ G_{12} = G_{13}, \quad G_{23} = \frac{E_2}{2(1+\nu_{23})} \end{gathered} \tag{1.82}$$

The compliance matrix can be written in terms of the 5 independent constants of a transverely isotropic material

$$[S] = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_1} & -\frac{\nu_{12}}{E_1} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{23}}{E_2} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2(1+\nu_{23})}{E_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{12}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \tag{1.83}$$

In the case of an isotropic material,

$$\begin{aligned} &E_1 = E_2 = E_3 = E, \quad \nu_{12} = \nu_{13} = \nu_{23} = \nu, \\ &G_{12} = G_{13} = G_{23} = G = \frac{E}{2(1+\nu)} \end{aligned} \tag{1.84}$$

$$[S] = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2(1+\nu)}{E} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2(1+\nu)}{E} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2(1+\nu)}{E} \end{bmatrix} \tag{1.85}$$

1.10 Representative material properties

In this course, we will consider laminated composite structures composed of carbon fiber-reinforced plies in all the exercises and assignments

1.10.1 Unidirectional carbon fiber-reinforced composite

and we will use a unidirectional carbon fiber-reinforced composite consisting of continuous IM7 fibers embedded in a 8552 epoxy matrix in the examples. The properties of the matrix, the fiber and the unidirectional fiber-reinforced composite are listed below.

The 8552 epoxy matrix is an isotropic material with the following properties

$$\rho_m = 1300 \text{ kg/m}^3, \quad E_m = 4.67 \text{ GPa}, \quad \nu_m = 0.37, \quad G_m = \frac{E}{2(1+\nu)} = 1.70 \text{ GPa} \tag{1.86}$$

where ρ_m is the mass density and the subscript m denotes matrix properties.

The IM7 carbon fibers are transversely isotropic with the following properties,

$$\begin{gathered}\rho_f = 1780 \text{ kg/m}^3, \quad E_{1f} = 276 \text{ GPa}, \quad E_{2f} = E_{3f} = 15 \text{ GPa}, \\ \nu_{12f} = \nu_{13f} = 0.29, \quad \nu_{23f} = 0.30, \\ G_{12f} = G_{13f} = 15 \text{ GPa}, \quad G_{23f} = \frac{E_{2f}}{2(1+\nu_{23f})} = 5.77 \text{ GPa}\end{gathered} \tag{1.87}$$

where the subscript f denotes fiber properties.

The effective properties of the IM7/8552 unidirectional fiber-reinforced composite are obtained using the asymptotic expansion homogenization (AEH) method. It uses a representative volume element with a hexagonal arrangement of IM7 fibers in a 8552 epoxy matrix. The AEH equations are solved numerically using the finite element method subject to periodic boundary conditions. The effective engineering elastic properties of the IM7/8552 unidirectional fiber-reinforced composite are listed below for a fiber volume fraction of 60%, i.e., $V_f = 0.6$.

$$\begin{gathered}\rho = 1588 \text{ kg/m}^3, \; E_1 = 167.4 \text{ GPa}, \quad E_2 = E_3 = 9.5 \text{ GPa}, \\ \nu_{12} = \nu_{13} = 0.33, \quad \nu_{23} = 0.44, \\ G_{12} = G_{13} = 4.8 \text{ GPa}, \quad G_{23} = \frac{E_2}{2(1+\nu_{23})} = 3.3 \text{ GPa}\end{gathered} \tag{1.88}$$

The strengths of IM7/8552 unidirectional fiber-reinforced composite are listed below

$$\begin{gathered}F_{1t} = 2,700 \text{ MPa}, \quad F_{1c} = 1,700 \text{ MPa}, \quad F_{2t} = 70 \text{ MPa}, \\ F_{2c} = 200 \text{ MPa}, \quad F_6 = 90 \text{ MPa}\end{gathered} \tag{1.89}$$

where F_{1t} is the longitudinal tensile strength, F_{1c} is the longitudinal compressive strength, F_{2t} is the transverse tensile strength, F_{2c} is the transverse compressive strength and F_6 is the in-plane shear strength.

1.10.2 Fabric-reinforced composite laminae

In some of the exercises, we will use fabric reinforced carbon/epoxy composite laminae whose representative engineering properties are listed below.

$$\rho = 1600 \text{ kg/m}^3, \; E_1 = 77.0 \text{ GPa}, \quad E_2 = 75.0 \text{ GPa}, \quad \nu_{12} = 0.06, \quad G_{12} = 6.5 \text{ GPa} \tag{1.90}$$

where the 1-axis is the warp direction and the 2-axis is the weft/fill direction.

The representative strengths of the fabric-reinforced carbon/epoxy composite laminae are listed below,

$$\begin{gathered} F_{1t} = 963 \text{ MPa}, \quad F_{1c} = 900 \text{ MPa}, \quad F_{2t} = 856 \text{ MPa}, \\ F_{2c} = 900 \text{ MPa}, \quad F_6 = 71 \text{ MPa} \end{gathered} \tag{1.91}$$

Exercises

Use the representative properties in Sec. 1.10.1 for unidirectional carbon/epoxy composites and the representative properties in Sec. 1.10.2 for fabric-reinforced carbon/epoxy composites.

1.1 Consider an x'-y'-z' coordinate system whose basis vectors $\mathbf{e}'_1, \mathbf{e}'_2$ and $\mathbf{e}'_3$ are oriented in the directions $\{1/\sqrt{2}, 1/\sqrt{2}, 0\}, \{-1/2, 1/2, 1/\sqrt{2}\}$ and $\{1/2, -1/2, 1/\sqrt{2}\}$, respectively, relative to the x-y-z coordinate system.

(a) Determine the stress transformation matrix $[T_\sigma]$.

(b) If at a point in a composite structure the stresses in the x-y-z coordinate system are

$$\begin{gathered} \sigma_x = 100 \text{ MPa}, \quad \sigma_y = -100 \text{ MPa}, \quad \sigma_z = 80 \text{ MPa}, \\ \tau_{yz} = 5 \text{ MPa}, \quad \tau_{xz} = 10 \text{ MPa}, \quad \tau_{xy} = 20 \text{ MPa} \end{gathered}$$

determine the stresses in the x'-y'-z' coordinate using the stress transformation matrix $[T_\sigma]$.

1.2 Consider an x'-y'-z' coordinate system whose basis vectors $\mathbf{e}'_1, \mathbf{e}'_2$ and $\mathbf{e}'_3$ are oriented in the directions $\{1/\sqrt{2}, 1/\sqrt{2}, 0\}, \{-1/2, 1/2, 1/\sqrt{2}\}$ and $\{1/2, -1/2, 1/\sqrt{2}\}$, respectively, relative to the x-y-z coordinate system.

(a) Determine the strain transformation matrix $[T_\varepsilon]$.

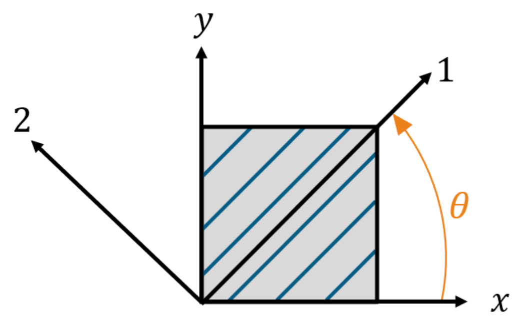
(b) If at a point in a composite structure the strains in the x-y-z coordinate system are

$$\begin{gathered} \varepsilon_x = 1000 \ \mu\varepsilon, \quad \varepsilon_y = -1500 \ \mu\varepsilon, \quad \varepsilon_z = 500 \ \mu\varepsilon, \\ \gamma_{yz} = 100 \ \mu\text{rad}, \quad \gamma_{xz} = 300 \ \mu\text{rad}, \quad \gamma_{xy} = 800 \ \mu\text{rad} \end{gathered}$$

determine the strains in the x'-y'-z' coordinate using the strain transformation matrix $[T_\varepsilon]$.

1.3 The homogenized (or effective) engineering properties of a unidirectional fiber-reinforced composite are listed in Sec. 1.10.1 for a fiber volume fraction $V_f = 0.6$. Determine the 6×6 elastic stiffness matrix $[C]$ of the unidirectional fiber-reinforced composite in GPa.

1.4 Consider the unidirectional fiber-reinforced composite whose elastic stiffness matrix $[C]$ was determined in Problem **1.3**. The fibers are oriented in the 1-direction as shown in the figure below where the angle $\theta = 45°$.



(a) Determine the components of the 6×6 elastic stiffness matrix (in GPa) in the x-y-z coordinate system.

(b) If the composite material is subjected to a stress $\sigma_x = 100$ MPa (all other stresses are zero), determine the strains in the x-y-z coordinate system in $\mu\varepsilon$. Is the shear strain $\gamma_{xy} = 0$? If not, provide a physical explanation for the shear strain when the material is subjected to a normal stress.

(c) If the composite material is subjected to stresses $\sigma_x = -100$ MPa and $\tau_{xy} = 50$ MPa (all other stresses are zero), determine the strain energy density U in kJ/m^3

Mechanics of a Lamina 2

2.1 Plane stress assumption

Laminated composite structures (or laminates) are typically thin-walled structures that are composed of multiple laminae. When a laminate is subjected to loads, the out-of-plane stress components τ_{13}, τ_{23} and σ_3 in the laminae are much smaller than the in-plane stress components σ_1, σ_2 and τ_{12}. We therefore neglect the out-of-plane stress components when analyzing thin-walled laminated composite structures. This is known as the plane stress assumption and can be formally stated as

$$\sigma_3 = \tau_{23} = \tau_{13} = 0 \tag{2.1}$$

The plane stress assumption greatly simplifies the analysis since we need to calculate only the in-plane stress components, namely the normal stresses σ_1, σ_2 and the shear stress τ_{12}, shown in Fig. 2.1.

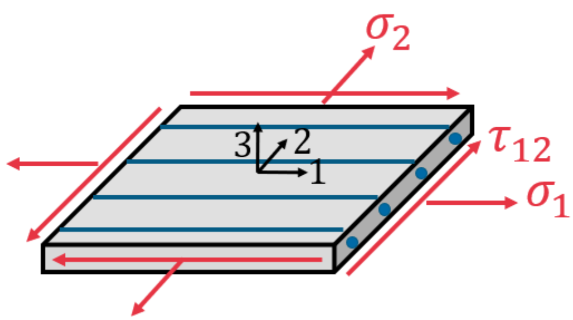


Figure 2.1: Stresses acting on a lamina in plane stress

The in-plane strain components are obtained by setting the out-of-plane stress components σ_3, σ_4 (i.e., τ_{23}), and σ_5 (i.e., τ_{13}) to zero in the right hand side of Eqn. (1.77)

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_1} & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} \tag{2.2}$$

where we have utilized the reciprocity relations (1.78) to relate the Poisson's ratio ν_{21} to ν_{12}, i.e., $\nu_{21} = \nu_{12}E_2/E_1$. In the plane stress assumption, the in-plane strains are related to the in-plane stresses through the longitudinal Young's modulus E_1, the in-plane transverse Young's modulus E_2, the in-plane Poisson's ratio ν_{12} and the in-plane shear modulus G_{12}.

It is important to note that although the transverse normal stress σ_3 is assumed to be zero, the transverse normal strain ε_3 need not be zero since the in-plane normal strains ε_1 and ε_2 can cause a Poisson's contraction/extension in the thickness direction. In fact, the transverse normal strain ε_3 can be obtained from (1.77) by setting out-of-plane stress components τ_{13} and τ_{23} and σ_3 to zero.

$$\varepsilon_3 = -\frac{\nu_{13}}{E_1}\sigma_1 - \frac{\nu_{23}}{E_2}\sigma_2 \tag{2.3}$$

A non-zero transverse normal strain ε_3 will cause the thickness of a laminated composite plane to either increase or decrease depending on its sign. This effect is known as thickness distention. Typically, we are primarily interested in the analysis of stress and failure rather than in the thickness distention of composite laminates. While we recognize that the transverse strain may be non-zero, we typically do not use Eqn. (2.3) in our analysis. If necessary, the transverse strain ε_3 can be calculated using (2.3) after the in-plane stress components σ_1 and σ_2 have been determined.

Equation (2.2) can be rewritten as follows,

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} \tag{2.4}$$

where the 3×3 matrix of compliances is the *reduced compliance matrix* $[S]$, and

$$S_{11} = \frac{1}{E_1}, \quad S_{12} = -\frac{\nu_{12}}{E_1}, \quad S_{22} = \frac{1}{E_2}, \quad S_{66} = \frac{1}{G_{12}} \tag{2.5}$$

The stress-strain relationship can be inverted

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix}^{-1} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} \tag{2.6}$$

and written in the following form

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} \tag{2.7}$$

where the 3×3 matrix of stiffnesses is the *reduced stiffness matrix* $[Q]$.

The following expressions for the reduced stiffness are obtained by inverting the 3×3 reduced compliance matrix analytically.

$$
\begin{aligned}
Q_{11} &= \frac{S_{22}}{S_{11}S_{22} - S_{12}^2}, \quad Q_{22} = \frac{S_{11}}{S_{11}S_{22} - S_{12}^2} \\
Q_{12} &= \frac{-S_{12}}{S_{11}S_{22} - S_{12}^2}, \quad Q_{66} = \frac{1}{S_{66}}
\end{aligned}
\tag{2.8}
$$

The reduced stiffness can be expressed in terms of the engineering properties by substituting for the compliance from Eq. (2.5) into Eq. (2.8). For example, the reduced stiffness Q_{11} has the following form,

$$
Q_{11} = \frac{S_{22}}{S_{11}S_{22} - S_{12}^2} = \frac{1/E_2}{\left(\frac{1}{E_1}\right)\left(\frac{1}{E_2}\right) - \left(-\frac{-\nu_{12}}{E_1}\right)\left(-\frac{\nu_{21}}{E_2}\right)} = \frac{E_1}{1 - \nu_{12}\nu_{21}} \tag{2.9}
$$

Similarly, we can express all the reduced stiffnesses in terms of the lamina engineering properties as follows

$$
\begin{aligned}
Q_{11} &= \frac{E_1}{1 - \nu_{12}\nu_{21}}, \quad Q_{22} = \frac{E_2}{1 - \nu_{12}\nu_{21}} \\
Q_{12} &= \frac{\nu_{12}E_2}{1 - \nu_{12}\nu_{21}}, \quad Q_{66} = G_{12}
\end{aligned}
\tag{2.10}
$$

It is important to note that the plane stress-reduced stiffnesses are not equal to the elastic stiffnesses, i.e., $Q_{ij} \neq C_{ij}$ and it is wrong to write the stress-strain relationship for plane stress as follows [2],

$$
\left\{\begin{matrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{matrix}\right\} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{66} \end{bmatrix} \left\{\begin{matrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{matrix}\right\} \tag{2.11}
$$

This is because the transverse normal strain ε_3 on the right hand side of Eqn. (1.43) does not equal zero for a lamina in plane stress.

2.2 Off-axis lamina

In general, fiber-reinforced laminated composite structures are made of multiple layers, each with its own specific fiber orientation. Typically, the fiber orientations are specified relative to fixed global structural coordinate system. The global or structural coordinate system is represented by x-y-z and the lamina principal material coordinate system is represented by 1-2-3 as shown in Fig. 2.2.

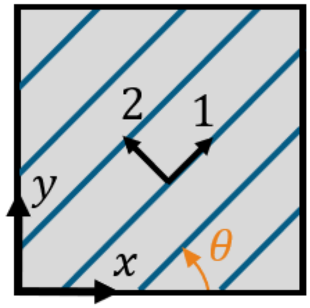


Figure 2.2: An off-axis lamina showing the global and principal material coordinate systems

2.2.1 Stress and strain transformation in 2D

When analyzing laminated composite structures, it is often necessary to transform stresses from the structural coordinate system to the lamina principal coordinate system and vice versa. For example, the stresses obtained using laminate analysis in a global structural coordinate system need to be transformed to the principal material coordinate system when performing failure analysis.

The 2D stress transformation relations for a lamina in plane stress are obtained from the 3D stress transformation relations by setting $\sigma_3 = \tau_{23} = \tau_{13} = 0$ in Eqn. (1.29),

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = [T_\sigma] \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}, \quad \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [T_\sigma]^{-1} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} \tag{2.12}$$

where in the case of plane stress, the stress transformation matrix (1.37) and its inverse reduce to

$$[T_\sigma] = \begin{bmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2 - n^2 \end{bmatrix}, \quad [T_\sigma]^{-1} = \begin{bmatrix} m^2 & n^2 & -2mn \\ n^2 & m^2 & 2mn \\ mn & -mn & m^2 - n^2 \end{bmatrix}, \tag{2.13}$$

Similarly, the following strain transformation relations for the in-plane strain components are obtained from Eqn. (1.33)

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} = [T_\varepsilon] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}, \quad \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = [T_\varepsilon]^{-1} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} \tag{2.14}$$

where the strain transformation matrix (1.38) and its inverse reduce to

$$[T_\varepsilon] = \begin{bmatrix} m^2 & n^2 & mn \\ n^2 & m^2 & -mn \\ -2mn & 2mn & m^2 - n^2 \end{bmatrix}, \quad [T_\varepsilon]^{-1} = \begin{bmatrix} m^2 & n^2 & -mn \\ n^2 & m^2 & mn \\ 2mn & -2mn & m^2 - n^2 \end{bmatrix} \tag{2.15}$$

2.2.2 Off-axis elastic stiffnesses

In this section, we derive the off-axis stiffness that relate the strains and stress in the global coordinate system. We begin with the stress-strain relationship (2.7) in the principal material coordinate system

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = [Q] \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} \tag{2.16}$$

The stresses and strains in the principal material coordinate system are expressed in terms of the stresses and strains in the global coordinate systems using Eqns. (2.12) and (2.14) to obtain

$$[T_\sigma] \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [Q][T_\varepsilon] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{2.17}$$

Next, we premultiply both sides of (2.17) by the inverse of the stress transformation matrix $[T_\sigma]$ to obtain

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [T_\sigma]^{-1}[Q][T_\varepsilon] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = [\overline{Q}] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{2.18}$$

where $[\overline{Q}] = [T_\sigma]^{-1}[Q][T_\varepsilon] = [T_\varepsilon]^T[Q][T_\varepsilon]$ is the *off-axis reduced stiffness matrix* that relate the stresses to the strains in the global coordinate system. Eqn. (2.18) can be written in the following form

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{2.19}$$

where $\overline{Q}_{ij}$ are the off-axis stiffnesses that are obtained by taking the product of $[T_\varepsilon]^T$, $[Q]$ and $[T_\varepsilon]$. The off-axis stiffnesses can expressed in terms of the stiffnesses in the principal material coordinate system and the angle θ as follows,

$$\begin{aligned}
\overline{Q}_{11} &= Q_{11}m^4 + 2(Q_{12} + 2Q_{66})m^2n^2 + Q_{22}n^4 \\
\overline{Q}_{12} &= (Q_{11} + Q_{22} - 4Q_{66})n^2m^2 + Q_{12}(n^4 + m^4) \\
\overline{Q}_{16} &= (Q_{11} - Q_{12} - 2Q_{66})nm^3 + (Q_{12} - Q_{22} + 2Q_{66})n^3m \\
\overline{Q}_{22} &= Q_{11}n^4 + 2(Q_{12} + 2Q_{66})n^2m^2 + Q_{22}m^4 \\
\overline{Q}_{26} &= (Q_{11} - Q_{12} - 2Q_{66})n^3m + (Q_{12} - Q_{22} + 2Q_{66})nm^3 \\
\overline{Q}_{66} &= (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})n^2m^2 + Q_{66}(n^4 + m^4)
\end{aligned} \tag{2.20}$$

where $m = \cos\theta$ and $n = \sin\theta$.

2.2.3 Off-axis elastic compliances

Next, we derive the off-axis compliances that relate the stresses and strains in the global coordinate system. We begin with the stress-strain relationship (2.4) in the principal material coordinate system

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} = [S] \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} \tag{2.21}$$

The strains and stresses in the principal material coordinate system are expressed in terms of the strains and stresses in the global coordinate systems using Eqns. (2.14) and (2.12) to obtain

$$[T_\varepsilon] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = [S][T_\sigma] \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \tag{2.22}$$

Next, both sides of (2.22) are premultiplied by the inverse of the strain transformation matrix $[T_\varepsilon]$ to obtain

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = [T_\varepsilon]^{-1}[S][T_\sigma] \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [\overline{S}] \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \tag{2.23}$$

where $[\overline{S}] = [T_\varepsilon]^{-1}[S][T_\sigma] = [T_\sigma]^T[S][T_\sigma]$ is the *off-axis reduced compliance matrix* that relates the strains to the strains in the global coordinate system. Eqn. (2.22) can be expressed in the following form

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \overline{S}_{11} & \overline{S}_{12} & \overline{S}_{16} \\ \overline{S}_{12} & \overline{S}_{22} & \overline{S}_{26} \\ \overline{S}_{16} & \overline{S}_{26} & \overline{S}_{66} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \tag{2.24}$$

where $\overline{S}_{ij}$ are the off-axis compliances that can be determined from the reduced compliances in the principal material coordinate system and the fiber orientation θ as follows

$$\begin{aligned}
\overline{S}_{11} &= S_{11}m^4 + (2S_{12}+S_{66})m^2n^2 + S_{22}n^4 \\
\overline{S}_{12} &= (S_{11}+S_{22}-S_{66})n^2m^2 + S_{12}(n^4+m^4) \\
\overline{S}_{16} &= (2S_{11}-2S_{12}-S_{66})nm^3 + (2S_{12}-2S_{22}+S_{66})n^3m \\
\overline{S}_{22} &= S_{11}n^4 + (2S_{12}+S_{66})n^2m^2 + S_{22}m^4 \\
\overline{S}_{26} &= (2S_{11}-2S_{12}-S_{66})n^3m + (2S_{12}-2S_{22}+S_{66})nm^3 \\
\overline{S}_{66} &= 2(2S_{11}+2S_{22}-4S_{12}-S_{66})n^2m^2 + S_{66}(n^4+m^4).
\end{aligned} \tag{2.25}$$

EXAMPLE 2.1

Consider an off-axis IM7-8552 unidirectional lamina that is subjected to a normal stres $\sigma_x = 100$ MPa. If $\theta = 30°$, determine the strains ε_x, ε_y and γ_{xy}.

$$\begin{aligned}
\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = [\overline{S}(30°)] \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} &= \begin{bmatrix} 48.26 & -19.44 & -63.16 \\ -19.44 & 97.91 & -22.83 \\ -63.16 & -22.83 & 138.47 \end{bmatrix} 10^{-12}\ \mathrm{Pa}^{-1} \cdot \begin{Bmatrix} 100 \\ 0 \\ 0 \end{Bmatrix} 10^6\ \mathrm{Pa} \\
&= \begin{Bmatrix} 4826 \\ -1944 \\ -6316 \end{Bmatrix} 10^{-6}
\end{aligned} \tag{2.26}$$

That is, the strains in the global coordinate system are,

$$\varepsilon_x = 4826\,\mu\varepsilon, \quad \varepsilon_y = -1944\,\mu\varepsilon, \quad \gamma_{xy} = -6316\,\mu\mathrm{rad} \tag{2.27}$$

It is observed that an off-axis lamina subjected to a normal stress σ_x will not only elongate in the x-direction and contract in the y-direction but it will also shear in the x-y plane as shown in Fig. 2.3 due to the non-zero shear strain γ_{xy}. This is referred as *extension-shear coupling*. The direction and magnitude of shear will depend on the fiber orientation θ.

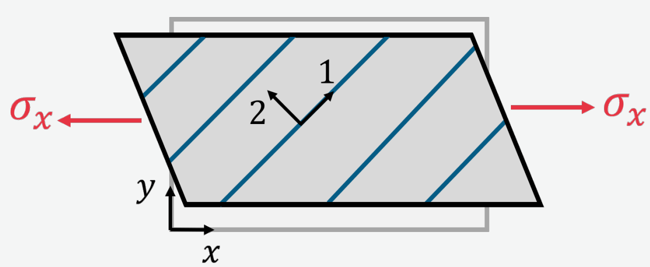


Figure 2.3: Response of an off-axis lamina to a normal stress σ_x

It can be similarly shown that an off-axis lamina that is subjected to a shear stress will exhibit normal strains due to shear-extension coupling.

2.3 Lamina analysis procedure

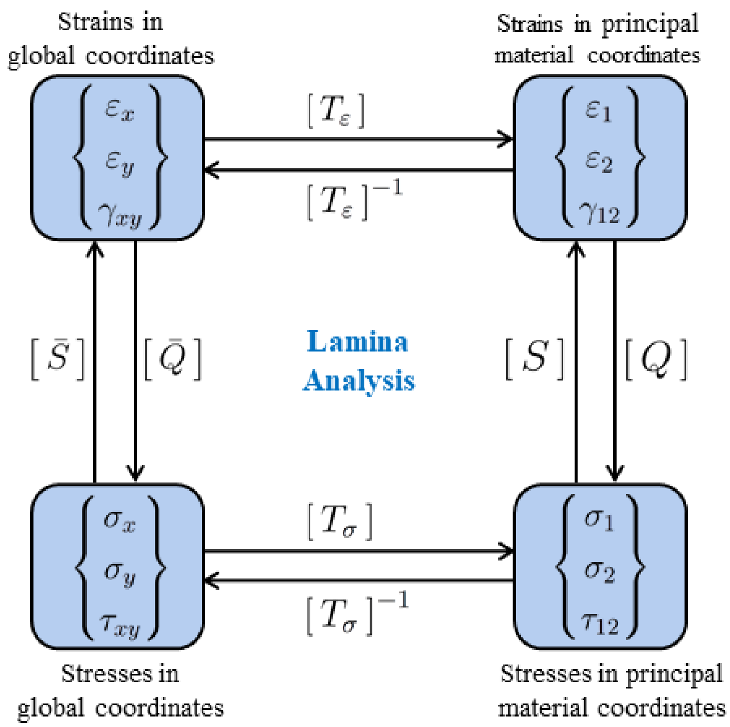


Figure 2.4: Concept map for the analysis of lamina. Adapted and modified from M.W. Hyer, Stress Analysis of Fiber-Reinforced Composite Materials, DEStech, 2009

2.4 Engineering properties of an off-axis lamina

In this section, we obtain the engineering properties of an off-axis lamina in a global coordinate system given the engineering properties (E_1, E_2, ν_{12} and G_{12}) in the principal material directions and the fiber orientation θ.

2.4.1 Young's modulus E_x and Poisson's ratio ν_{xy}

To determine the Young's modulus E_x and the Poisson's ratio ν_{xy} in the global coordinate system, we apply a normal stress σ_x with $\sigma_y = \tau_{xy} = 0$. The resulting normal strain ε_x is determined from the stress-strain relations (2.24) as

$$\varepsilon_x = \overline{S}_{11}\sigma_x \tag{2.28}$$

The Young's modulus E_x in the global coordinate system is defined as the ratio of the applied normal stress σ_x to the resulting normal strain ε_x, i.e.,

$$E_x = \frac{\sigma_x}{\varepsilon_x} = \frac{1}{\overline{S}_{11}} \tag{2.29}$$

The off-axis compliance $\overline{S}_{11}$ can be expressed in terms of the engineering properties in the principal material coordinate system and the fiber orientation θ as,

$$\begin{aligned} \overline{S}_{11} &= S_{11}m^4 + (2S_{12} + S_{66})m^2n^2 + S_{22}n^4 \\ &= \frac{1}{E_1}m^4 + \left(-2\frac{\nu_{12}}{E_1} + \frac{1}{G_{12}}\right)m^2n^2 + \frac{1}{E_2}n^4 \end{aligned} \tag{2.30}$$

Substituting for the off-axis compliance $\overline{S}_{11}$ from (2.30) into (2.29) give the following expression for the off-axis Young's modulus E_x,

$$E_x = \frac{E_1}{m^4 + \left(\frac{E_1}{G_{12}} - 2\nu_{12}\right)m^2n^2 + \frac{E_1}{E_2}n^4} \tag{2.31}$$

Next, the off-axis Poisson's ratio ν_{xy} is obtained by taking the negative of the ratio of the transverse normal strain and the longitudinal normal strain when an off-axis lamina is subjeced to a longitudinal normal stress.

When the off-axis lamina is subjected to a normal stress σ_x, the resulting transverse normal strain ε_y is obtained using the stress-strain relations (2.24) as

$$\varepsilon_y = \overline{S}_{12}\sigma_x \tag{2.32}$$

The off-axis Poisson's ratio is determined by taking the ratio of the transverse normal strain ε_y and the longitudinal normal strain ε_x in Eqns. (2.32) and (2.28) ,

$$\nu_{xy} = -\frac{\varepsilon_y}{\varepsilon_x} = -\frac{\overline{S}_{12}\sigma_x}{\overline{S}_{11}\sigma_x} = -\frac{\overline{S}_{12}}{\overline{S}_{11}} \tag{2.33}$$

The off-axis Poisson's ratio ν_{xy} can be expressed in terms of the engineering properties in the principal material coordinate system and the fiber orientation θ using Eqns. (2.25) and (2.5),

$$\nu_{xy} = \frac{\nu_{12}\left(m^4 + n^4\right) - \left(1 + \frac{E_1}{E_2} - \frac{E_1}{G_{12}}\right) m^2 n^2}{m^4 + \left(\frac{E_1}{G_{12}} - 2\nu_{12}\right) m^2 n^2 + \frac{E_1}{E_2} n^4} \tag{2.34}$$

2.4.2 Young's modulus E_y

The Young's modulus E_y in the global coordinate system can be obtained by applying a stress σ_y with $\sigma_x = \tau_{xy} = 0$.

$$E_y = \frac{\sigma_y}{\varepsilon_y} = \frac{\sigma_y}{\overline{S}_{22}\sigma_y} = \frac{1}{\overline{S}_{22}} \tag{2.35}$$

The off-axis Young's modulus E_y can be expressed in terms of the engineering properties in the principal material coordinate system and the fiber orientation θ as follows

$$E_y = \frac{E_2}{m^4 + \left(\frac{E_2}{G_{12}} - 2\nu_{12}\right) m^2 n^2 + \frac{E_2}{E_1} n^4} \tag{2.36}$$

2.4.3 Shear modulus G_{xy}

The off-axis shear modulus G_{xy} is obtained by applying a shear stress τ_{xy} with the normal stresses being zero ($\sigma_x = \sigma_y = 0$). The shear strain γ_{xy}, obtained from the stress-strain relations (2.24), is

$$\gamma_{xy} = \overline{S}_{66}\tau_{xy} \tag{2.37}$$

The off-axis shear modulus G_{xy} is defined as the ratio of the applied shear stress τ_{xy} to the resulting shear strain γ_{xy}. That is,

$$G_{xy} = \frac{\tau_{xy}}{\gamma_{xy}} = \frac{\tau_{xy}}{\overline{S}_{66}\tau_{xy}} = \frac{1}{\overline{S}_{66}} \tag{2.38}$$

The off-axis shear modulus can be expressed in terms of the engineering properties in the principal material coordinate system and the fiber orientation θ using Eqns. (2.25) and (2.5) , as follows

$$G_{xy} = \frac{G_{12}}{m^4 + n^4 + 2m^2 n^2 \left[\frac{2G_{12}}{E_1}\left(1 + 2\nu_{12}\right) + \frac{2G_{12}}{E_2} - 1\right]} \tag{2.39}$$

EXAMPLE 2.2

The variation of off-axis engineering properties with angle θ is shown in Fig. 2.5 for IM7-8552 unidirectional carbon fiber-reinforced lamina.

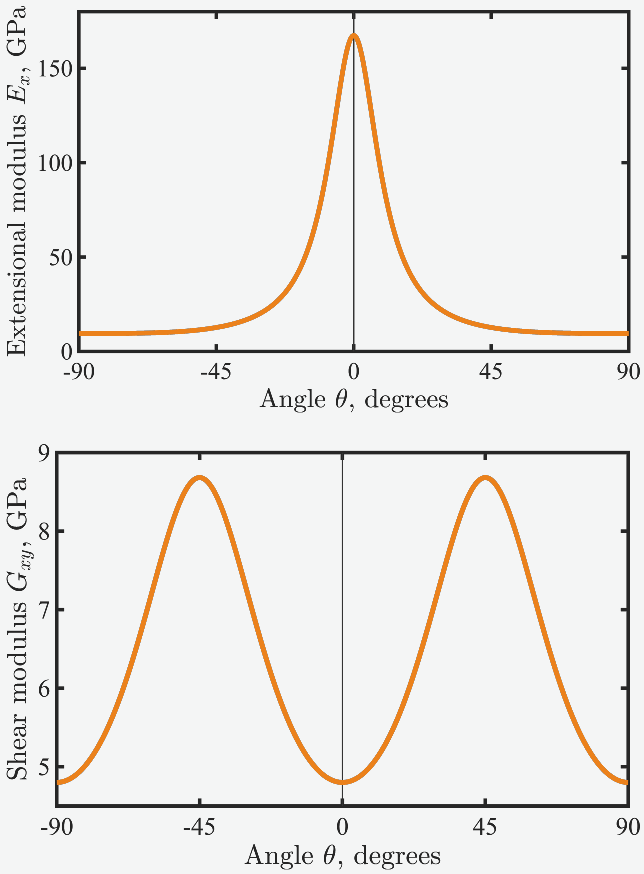


Figure 2.5: Variation of engineering properties with fiber angle θ for a carbon fiber-reinforced composite lamina

2.5 Tsai-Wu failure theory

When designing laminated composite structures, we need to make sure that the structure can withstand the applied loads. When analyzing a composite structure, we first determine the stresses in each lamina and then use a failure theory to determine the factor of safety. Several failure theories have been proposed for composite materials. Here, we will use the Tsai-Wu failure theory.

2.5.1 Failure criterion

The Tsai-Wu failure theory postulates that failure will occur when

$$g(\sigma_1, \sigma_2, \tau_{12}) = f_1\sigma_1 + f_2\sigma_2 + f_6\tau_{12} + f_{11}\sigma_1^2 + f_{22}\sigma_2^2 + f_{66}\tau_{12}^2 + 2f_{12}\sigma_1\sigma_2 + 2f_{16}\sigma_1\tau_{12} + 2f_{26}\sigma_2\tau_{12} = 1 \quad (2.40)$$

where, $f_1, \ldots, f_{26}$ are the Tsai-Wu failure coefficients and $\sigma_1, \sigma_2, \tau_{12}$ are the stresses in the principal material coordinate system. The lamina will not fail if the left hand side is less than 1.

2.5.2 Determining the failure coefficients

It can be systematically shown that f_6, f_{16} and f_{26} are zero. Thus, failure will occur when

$$g(\sigma_1, \sigma_2, \tau_{12}) = f_1\sigma_1 + f_2\sigma_2 + f_{11}\sigma_1^2 + f_{22}\sigma_2^2 + f_{66}\tau_{12}^2 + 2f_{12}\sigma_1\sigma_2 = 1 \quad (2.41)$$

The failure coefficient f_{12} needs to be obtained experimentally using biaxial loading tests. In the absence of experimental data, it is normally assumed that $f_{12} = -\frac{1}{2}\sqrt{f_{11}f_{22}}$. The reduced Tsai-Wu equation is

$$f_1\sigma_1 + f_2\sigma_2 + f_{11}\sigma_1^2 + f_{22}\sigma_2^2 + f_{66}\tau_{12}^2 - \sqrt{f_{11}f_{22}}\,\sigma_1\sigma_2 = 1 \quad (2.42)$$

The Tsai-Wu coefficients can be determined by applying the failure theory to uniaxial loading cases. Since the Tsai-Wu failure criteria needs to be satisfied at failure, we obtain a system of equations that can be solved to obtain the following expressions for the Tsai-Wu coefficients in terms of the strengths,

$$f_1 = \frac{1}{F_{1t}} - \frac{1}{F_{1c}}, \quad f_{11} = \frac{1}{F_{1t}F_{1c}}, \quad f_2 = \frac{1}{F_{2t}} - \frac{1}{F_{2c}}, \quad f_{22} = \frac{1}{F_{2t}F_{2c}}, \quad f_{66} = \frac{1}{F_6^2} \quad (2.43)$$

where F_{1t} is the longitudinal tensile strength, F_{1c} is the longitudinal compressive strength, F_{2t} is the transverse tensile strength, F_{2c} is the transverse compressive strength and F_6 is the in-plane shear strength of the lamina in the principal material coordinate system.

2.5.3 Calculating the factor of safety using the Tsai-Wu failure criterion

Given a stress state $(\sigma_1, \sigma_2, \tau_{12})$, the safety factor S_f is a stress multiplier that when applied to all stress components will cause the material to fail. That is, failure will initiate when the stress state is $(S_f\sigma_1, S_f\sigma_2, S_f\tau_{12})$. Substituting into the Tsai-Wu failure criterion gives,

$$f_1\left(S_f\sigma_1\right) + f_2\left(S_f\sigma_2\right) + f_{11}\left(S_f\sigma_1\right)^2 + f_{22}\left(S_f\sigma_2\right)^2 + f_{66}\left(S_f\tau_{12}\right)^2 - \sqrt{f_{11}f_{22}}\,\left(S_f\sigma_1\right)\left(S_f\sigma_2\right) = 1 \quad (2.44)$$

This is a quadratic equation for the factor of safety S_f, which can be written as follows

$$aS_f^2 + bS_f - 1 = 0 \tag{2.45}$$

where

$$\begin{aligned} a &= f_{11}\sigma_1^2 + f_{22}\sigma_2^2 + f_{66}\tau_{12}^2 - \sqrt{f_{11}f_{22}}\,\sigma_1\sigma_2, \\ b &= f_1\sigma_1 + f_2\sigma_2. \end{aligned} \tag{2.46}$$

It is noted that the coefficients a and b are dimensionless values. The quadratic equation (2.45) yields two roots for S_f. The positive root, which is denoted as S_{fa}, is the factor of safety for the actual stress state

$$S_{fa} = \frac{-b + \sqrt{b^2 + 4a}}{2a} \tag{2.47}$$

The quadratic equation (2.45) also yields a negative root, denoted by S_{fr}, which is the hypothetical factor of safety when the the signs of all three stress components are reversed. The factor of safety S_{fr} corresponds to a situation when the loads are reversed thereby causing the stress components to change sign.

$$S_{fr} = \frac{-b - \sqrt{b^2 + 4a}}{2a} \tag{2.48}$$

EXAMPLE 2.3

Determine the factor of safety of an IM7-8552 unidirectional off-axis lamina that is oriented at $\theta = 30°$ and subjected to the stress state shown in Fig. 2.6.

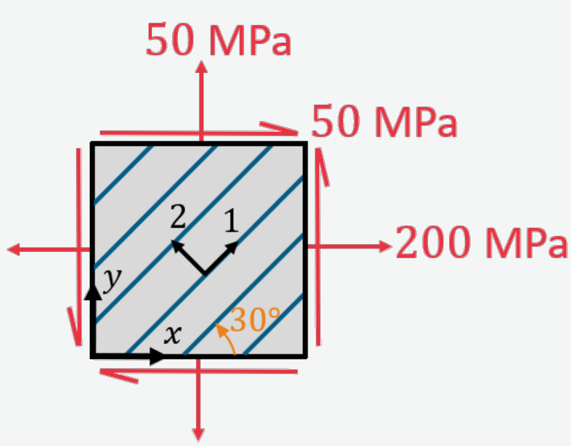


Figure 2.6: Stresses in the principal material coordinate system

In order to analyze the failure of the lamina, we need to first determine the stresses in the principal material coordinate system using the stress transformation matrix.

$$\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \tau_{12}\end{Bmatrix} = [T_\sigma(30^\circ)]\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix} = \begin{bmatrix}3/4 & 1/4 & \sqrt{3}/2\\ 1/4 & 3/4 & -\sqrt{3}/2\\ -\sqrt{3}/4 & \sqrt{3}/4 & 1/2\end{bmatrix}\begin{Bmatrix}200.0\\ 50.0\\ 50.0\end{Bmatrix} = \begin{Bmatrix}205.8\\ 44.20\\ -39.95\end{Bmatrix}\text{MPa} \tag{2.49}$$

Thus, the stresses in the principal material coordinate system are $\sigma_1 = 205.8$ MPa, $\sigma_2 = 44.20$ MPa and $\tau_{12} = -39.95$ MPa as shown in Fig. 2.7.

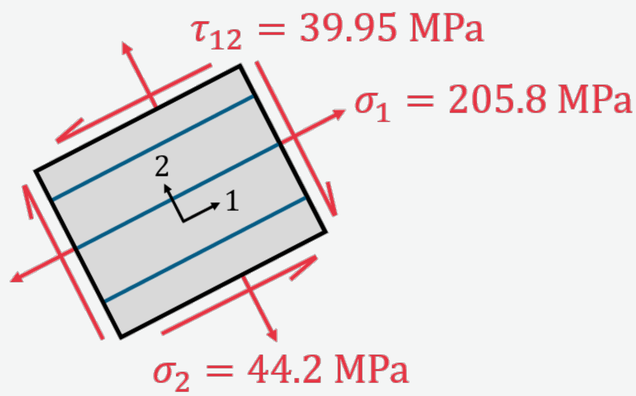


Figure 2.7: Stresses in the principal material coordinate system

Next, the Tsai-Wu failure coefficients are determined using Eqn. (2.43) and the lamina strengths listed in Eqn. (1.89).

$$\begin{gathered} f_1 = -2.179\times10^{-10}\ \text{Pa}^{-1},\quad f_{11} = 2.179\times10^{-19}\ \text{Pa}^{-2},\quad f_2 = 9.286\times10^{-9}\ \text{Pa}^{-1}\\ f_{22} = 7.143\times10^{-17}\ \text{Pa}^{-2},\quad f_{66} = 1.235\times10^{-16}\ \text{Pa}^{-2}\end{gathered} \tag{2.50}$$

The coefficients a and b are determined from Eqn. (2.46),

$$\begin{aligned} a &= f_{11}\sigma_1^2 + f_{22}\sigma_2^2 + f_{66}\tau_{12}^2 - \sqrt{f_{11}f_{22}}\,\sigma_1\sigma_2 = 0.310\\ b &= f_1\sigma_1 + f_2\sigma_2 = 0.366\end{aligned} \tag{2.51}$$

Next, the factor of safety S_{fa} for the actual stress state is determined using Eqn. (2.47),

$$S_{fa} = \frac{-b+\sqrt{b^2+4a}}{2a} = 1.30 \tag{2.52}$$

In this example, the transverse normal stress σ_2 is tensile. Since a unidirectional fiber reinforced composite has a low tensile strength F_{2t} in the transverse direction, the transverse tensile stress contributes to the low factor of safety.

The factor of safety S_{fr} for a reversed-in-sign state of stress is

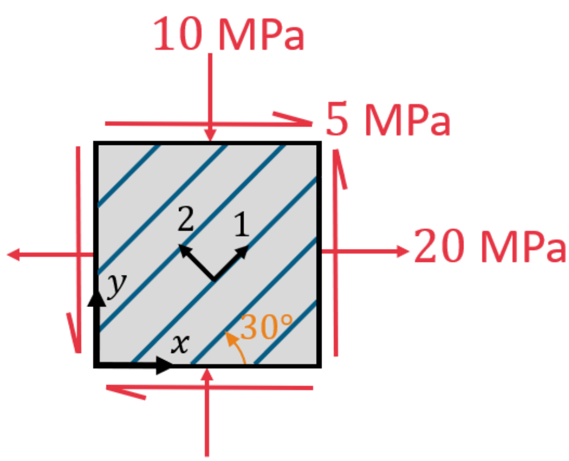
$$S_{fr} = \frac{-b - \sqrt{b^2 + 4a}}{2a} = -2.48 \tag{2.53}$$

The magnitude of the factor of safety S_{fr} is larger than S_{fa} due to the compressive transverse normal stress σ_2 when the stresses are reversed.

Exercises

Use the representative properties in Sec. 1.10.1 for unidirectional carbon/epoxy composites and the representative properties in Sec. 1.10.2 for fabric-reinforced carbon/epoxy composites. Use the Tsai-Wu theory for failure analysis.

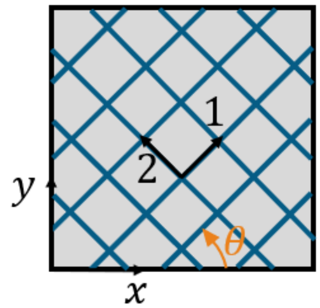
2.1 Consider an off-axis unidirectional carbon fiber-reinforced composite lamina with the fibers oriented at $\theta = 60°$ and subjected to the stress state shown in the figure below.



Determine,

(a) the off-axis compliance matrix $[\bar{S}(60°)]$ and the stiffness matrix $[\bar{Q}(60°)]$.
(b) the strains ε_x, ε_y and γ_{xy} in the global coordinate system.
(c) the strains ε_1, ε_2 and γ_{12} in the principal material coordinate system.

2.2 Consider a woven carbon fabric-reinforced composite lamina whose 1-direction (i.e., the warp direction) is oriented at an angle of θ relative to the x-axis.



(a) Plot the off-axis engineering properties properties E_x, E_y, G_{xy} and ν_{xy} as a function of the fiber orientation θ over the range $-90°$ to $90°$. Calculate the percent change in G_{xy} and ν_{xy} for a fiber orientation of $45°$ compared to $0°$.

(b) Plot the off-axis stiffness $\bar{Q}_{16}$ and compliances $\bar{S}_{11}$ and $\bar{S}_{66}$ as a function of the fiber orientation θ.

(c) Do the variation of the engineering properties and compliances make sense? Explain your reasoning.

2.3 Consider a tensile specimen of a unidirectional carbon fiber-reinforced composite with a rectangular cross section of width 25 mm and thickness 4 mm. The fibers are oriented at $\theta = 30°$ to the longitudinal edge and the specimen is subjected to an axial force of 10 kN.

(a) Determine the factor of safety S_{fa}. What is the maximum tensile force that can be applied to the specimen?

(b) Calculate the factor of safety S_{fr}. What does the magnitude of S_{fr} tell us in this application? Why are the magnitudes of S_{fa} and S_{fr} different?

(c) Obtain the factor of safety S_{fa} when the fibers are oriented at $\theta = 60°$ to the longitudinal edge of the specimen. Will the specimen be able to withstand the load?

Part II: The Classical Lamination Theory and its Applications

Classical Laminated Plate Theory 3

3.1 Kinematics of deformation: The Kirchhoff Hypothesis

Consider an N-layer laminated plate that is initially flat as shown in Figure 3.1. A global x-y-z Cartesian coordinate system is introduced in which the x-y plane coincides with the geometric mid-surface of the plate. The laminated plate is subjected to loads and it's deformation is quantified by the displacements u, v and w of each point in the x, y and z direction, respectively.

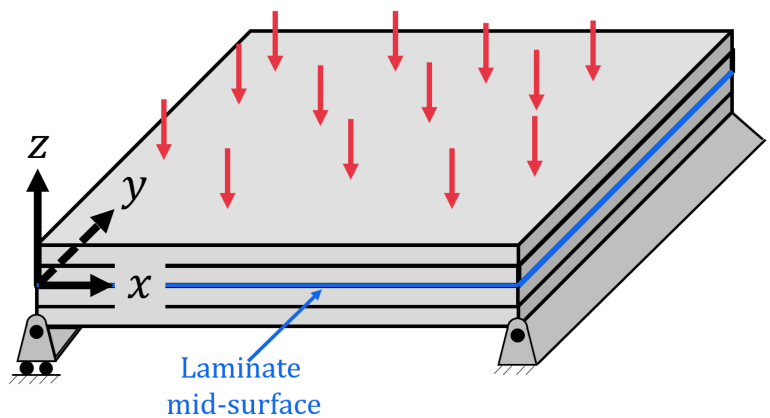


Figure 3.1: Schematic of a laminated composite plate

The layers or laminae are numbered from 1 to N starting from the bottom as shown in Figure 3.2. The z coordinate specifies the location of a point in the thickness direction relative to the mid-surface.

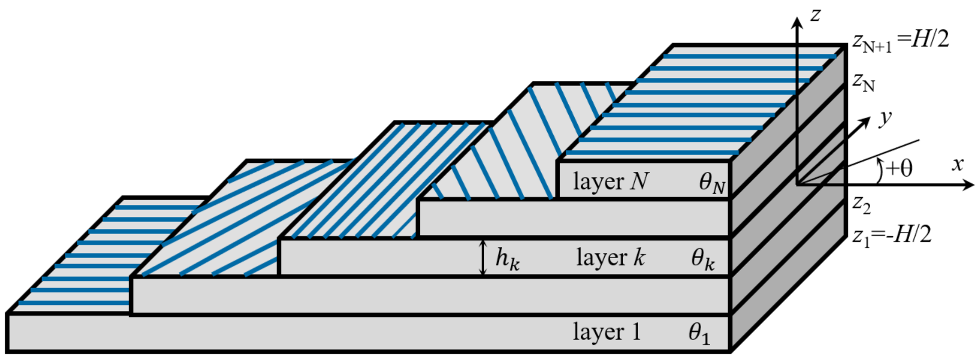


Figure 3.2: Section of a laminate that shows the numbering and orientation of the laminae

The k^{th} lamina extends from z_k to z_{k+1} in the thickness direction and has a fiber orientation of θ_k relative to the x-axis. The thickness of the k^{th} lamina is $h_k = z_{k+1} - z_k$ and the total thickness of the laminate is H.

The deformation of the laminated plate is analyzed using the *Classical Laminated Plate Theory* (CLPT) which is based on the Kirchhoff hypothesis. The fundamental assumptions of CLPT are:

1. The displacements are small compared to the thickness of the plate.
2. Material line elements that are straight and perpendicular to the mid-surface before deformation can rotate but they remain straight and normal to the mid-surface after deformation as shown in Fig. 3.3
3. The length of material line elements that are perpendicular to the mid-surface remain unchanged.

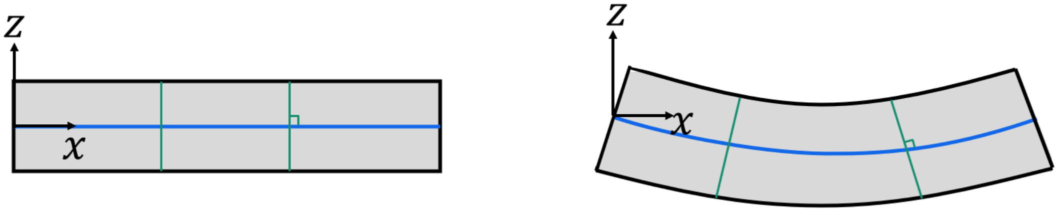


Figure 3.3: Kirchhoff assumption for the deformation of a classical laminated plate

Since the displacements are small in keeping with Assumption 1, the intensity of deformation is characterized by the infinitesimal strains. The corresponding strain-displacement relations are

$$\begin{gathered}\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \frac{\partial w}{\partial z} \\ \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\end{gathered} \tag{3.1}$$

Assumption 3 implies that the transverse normal strain ε_z is zero, i.e.,

$$\varepsilon_z = \frac{\partial w}{\partial z} = 0 \tag{3.2}$$

which when integrated with respect to z gives the following general form for the transverse displacement,

$$w = w_o(x, y, t) \tag{3.3}$$

The general form for w indicates that the transverse displacement can vary as a function of the in-plane coordinates x, y, and time t, but it is independent of the z coordinate. In other words, all points through the thickness of a laminated plate experience the same transverse displacement $w_o(x, y, t)$ under externally applied loads. This is a direct consequence of the Kirchhoff assumption that the length of the transverse normals remain unchanged.

Assumption 2 implies that the transverse shear strains are zero. Equation (3.1) in conjunction with (3.3) gives,

$$\begin{aligned} \gamma_{xz} &= \frac{\partial u}{\partial z} + \frac{\partial w_o}{\partial x} = 0 \\ \frac{\partial u}{\partial z} &= -\frac{\partial w_o}{\partial x} \end{aligned} \tag{3.4}$$

which when integrated with respect to z gives a general form for the displacement in the x direction

$$u = -z\frac{\partial w_o}{\partial x} + u_o(x, y, t) \tag{3.5}$$

where $u_o(x, y, t)$ is an arbitrary function of the in-plane coordinates x,y and time t.

Similarly, by setting the transverse shear strain γ_{yz} in (3.1) to zero, we obtain the following general form for the displacement $v(x, y, t)$,

$$v = -z\frac{\partial w_o}{\partial y} + v_o(x, y, t) \tag{3.6}$$

where $v_o(x, y, t)$ is an arbitrary function of the in-plane coordinates x,y and time t.

In summary, the general forms of the displacements based on the Kirchhoff assumptions can be expressed as follows

$$\begin{aligned} u(x, y, z, t) &= u_o(x, y, t) - z\frac{\partial w_o(x, y, t)}{\partial x} \\ v(x, y, z, t) &= v_o(x, y, t) - z\frac{\partial w_o(x, y, t)}{\partial y} \\ w(x, y, z, t) &= w_o(x, y, t) \end{aligned} \tag{3.7}$$

The deformation kinematics of the classical laminated plate theory is illustrated in Figure 3.4. Let's consider a material line element AB that is initially perpendicular to the mid-surface. After the loads are applied, segment AB rotates in the x-z plane but remains normal to the deformed mid-surface. The slope of the deformed mid-surface is denoted by the angle α_x where

$$\alpha_x = \frac{\partial w_o}{\partial x} \tag{3.8}$$

Since segment AB remains perpendicular to the deformed mid-surface (Assumption 2), it rotates counterclockwise by an angle α_x. If we consider point C that is located at distance of z from the mid-surface, its horizontal displacement equals the horizontal displacement of the mid-surface minus a displacement in the negative x-direction due to the rotation of the normal. In other words,

$$u = u_o - z\alpha_x = u_o - z\frac{\partial w_o}{\partial x} \tag{3.9}$$

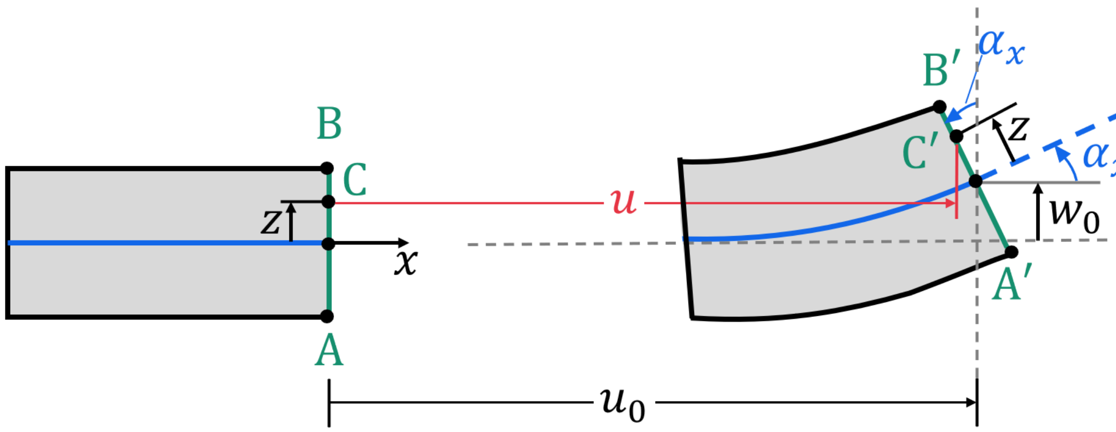


Figure 3.4: Kinematics of deformation based on the Kirchhoff Hypothesis

3.2 Laminate strains

The following expressions for the in-plane strains in a laminated composite plate are obtained by substituting the displacements u, v and w from Eqn. (3.7) into the strain-displacement relations (3.1)

$$
\begin{aligned}
\varepsilon_x &= \frac{\partial u}{\partial x} = \frac{\partial u_o}{\partial x} - z\frac{\partial^2 w_o}{\partial x^2} \\
\varepsilon_y &= \frac{\partial v}{\partial y} = \frac{\partial v_o}{\partial y} - z\frac{\partial^2 w_o}{\partial y^2} \\
\gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \left(\frac{\partial u_o}{\partial y} - z\frac{\partial^2 w_o}{\partial y \partial x}\right) + \left(\frac{\partial v_o}{\partial x} - z\frac{\partial^2 w_o}{\partial x \partial y}\right) \\
&= \left(\frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x}\right) - 2z\frac{\partial^2 w_o}{\partial x \partial y}
\end{aligned}
\tag{3.10}
$$

The in-plane strain components can be expressed in the following array form

$$
\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} + z \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \tag{3.11}
$$

where ε_x^o, ε_y^o and γ_{xy}^o are the mid-surface strains,

$$
\varepsilon_x^o = \frac{\partial u_o}{\partial x}. \quad \varepsilon_y^o = \frac{\partial v_o}{\partial y}, \quad \gamma_{xy}^o = \frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x} \tag{3.12}
$$

and κ_x, κ_y and κ_{xy} are the mid-surface curvatures,

$$
\kappa_x = -\frac{\partial^2 w_o}{\partial x^2}, \quad \kappa_y = -\frac{\partial^2 w_o}{\partial y^2}, \quad \kappa_{xy} = -2\frac{\partial^2 w_o}{\partial x \partial y} \tag{3.13}
$$

It is noted from (3.11) that the strain components have a linear variation in the thickness direction. The strains ε_x^o, ε_y^o are the normal strains and γ_{xy}^o is the in-plane shear strain experienced by an element on the mid-surface at $z = 0$. The quantities κ_x and κ_y are the curvatures of the mid-surface in the x- and y-directions, respectively. The shapes of the deformed mid-surface corresponding to positive curvatures κ_x and κ_y are shown in Fig. 3.5.

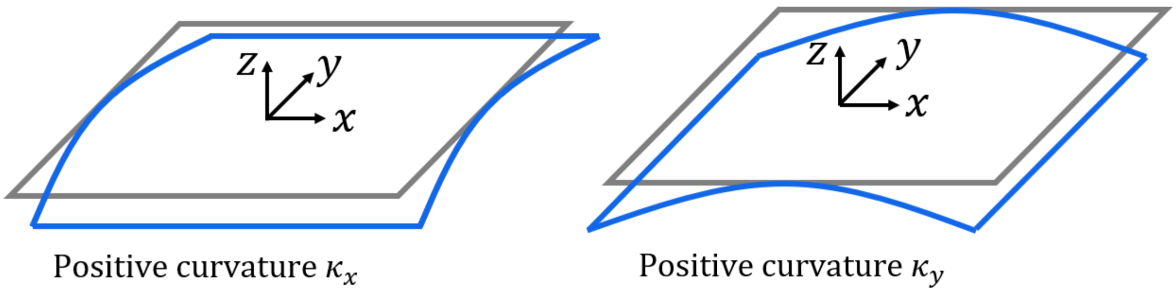


Figure 3.5: Mid-surface curvatures in the x and y directions

The quantity κ_{xy} is a twisting curvature. The shape of the mid-surface corresponding to a positive κ_{xy} is depicted in Fig. 3.6.

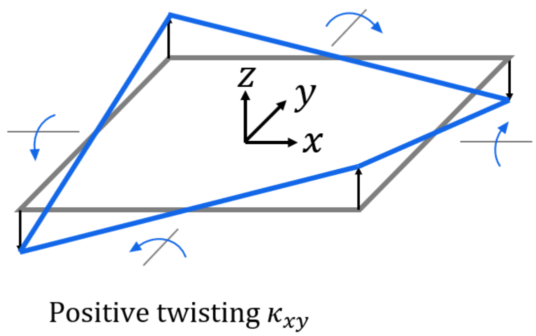


Figure 3.6: Mid-surface twisting curvature

3.3 Laminate stresses

The strains at any location within a laminate can be calculated using Eqn. (3.11) if the mid-surface strain ε_x^o, ε_y^o and γ_{xy}^o and curvatures κ_x, κ_y and κ_{xy} are known. Since every point within the laminate is assumed to be in a state of plane stress, the stresses at a distance z from the mid-surface can be determined from the strains at that location using the plane stress reduced constitutive equations (2.19) for an off-axis ply.

$$\begin{Bmatrix} \sigma_x(z) \\ \sigma_y(z) \\ \tau_{xy}(z) \end{Bmatrix} = [\overline{Q}(z)] \begin{Bmatrix} \varepsilon_x(z) \\ \varepsilon_y(z) \\ \gamma_{xy}(z) \end{Bmatrix} \tag{3.14}$$

where $[\overline{Q}(z)]$ is the reduced stiffness matrix at z. The in-plane stresses can be directly related to the mid-surface strains and curvatures by substituting for the strain from (3.11) into (3.14)

$$\begin{Bmatrix} \sigma_x(z) \\ \sigma_y(z) \\ \tau_{xy}(z) \end{Bmatrix} = [\overline{Q}(z)] \left(\begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} + z \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} \right) \tag{3.15}$$

The reduced stiffnesses $\overline{Q}_{ij}$ depend on the lamina fiber orientation θ. In the case of a laminated composite plate, each lamina has its own fiber orientation θ_k where the subscript k denotes the layer number. The fiber orientation and stiffnesses are assumed to be constant with each lamina. Therefore, the reduced stiffnesses at a point whose thickness coordinate is z in layer k can be written as $\overline{Q}_{ij}(z) = \overline{Q}_{ij}^{(k)}$ where $\overline{Q}_{ij}^{(k)}$ are the reduced stiffnesses for the k^{th} lamina. The in-plane stresses at that point are,

$$\begin{Bmatrix} \sigma_x(z) \\ \sigma_y(z) \\ \tau_{xy}(z) \end{Bmatrix}^{(k)} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix}^{(k)} \left(\begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} + z \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \right) \tag{3.16}$$

Since the reduced stiffnesses have a piecewise constant variation and the strains have a linear variation in the thickness direction, the *stresses exhibit a piecewise linear variation* through the thickness of the laminate.

3.4 Force and moment resultants

When analyzing laminated composite plates, the net loads acting on an element are represented by force and moment resultants. The in-plane force resultants are defined as the resultant forces per width and are obtained by integrating the stresses through the thickness of the laminate.

$$\begin{aligned} N_x &= \int_{-H/2}^{H/2} \sigma_x\,(z)\,dz \\ N_y &= \int_{-H/2}^{H/2} \sigma_y\,(z)\,dz \\ N_{xy} &= \int_{-H/2}^{H/2} \tau_{xy}\,(z)\,dz \end{aligned} \tag{3.17}$$

The in-plane force resultants are by definition forces per unit width and have units of N/m The force resultants N_x and N_y represent the normal forces acting on the element in the x- and y-directions, respectively. The force resultant N_{xy} represents the shear force acting parallel to the edges. The force resultants are distributed forces acting the edges of an element as illustrated in Fig. 3.7(a) although they are usually represented by single arrows as shown in Fig. 3.7(b) for the sake of convenience.

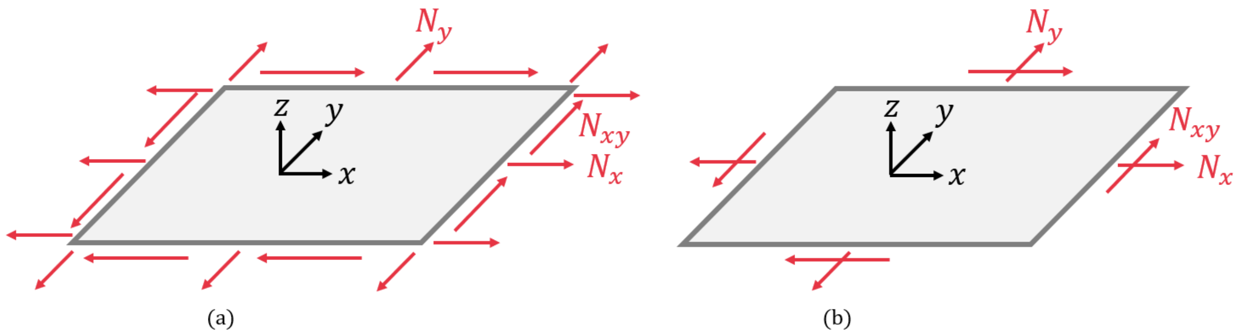


Figure 3.7: Force resultants acting on an element

The three integrals in (3.17) can be represented an integral of a column array of in-plane stresses,

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \int_{-H/2}^{H/2} \begin{Bmatrix} \sigma_x(z) \\ \sigma_y(z) \\ \tau_{xy}(z) \end{Bmatrix} dz \tag{3.18}$$

The moment resultants are defined as

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \int_{-H/2}^{H/2} \begin{Bmatrix} \sigma_x(z) \\ \sigma_y(z) \\ \tau_{xy}(z) \end{Bmatrix} z\, dz \tag{3.19}$$

where M_x and M_y are the *bending* moments and M_{xy} is the *twisting* moment that act on the edges of element as shown in Fig. 3.8. The moments resultants are by definition moments per unit width and have units of N·m/m.

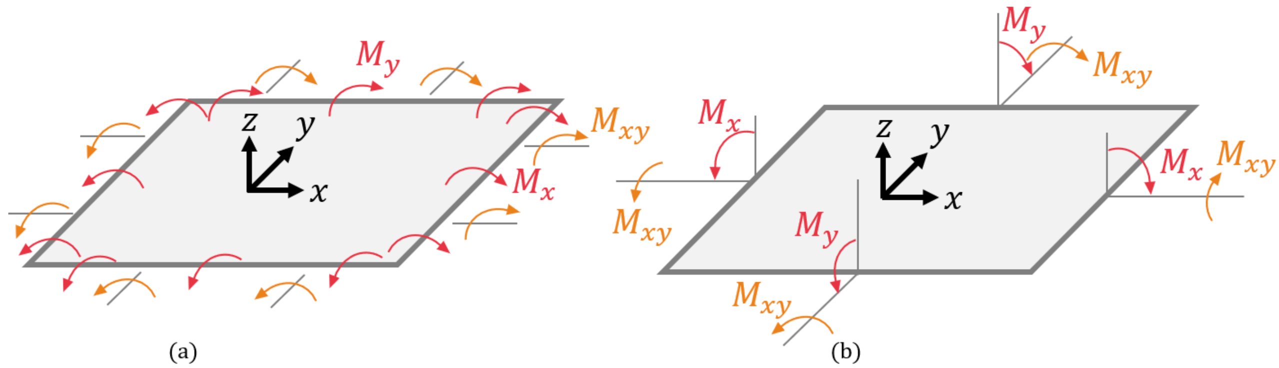


Figure 3.8: Moment resultants acting on an element

3.5 Load-deformation relations

In this section, we relate the force and moment resultants acting on an element to its mid-surface strains and curvatures. The stresses from (3.15) are substituted into the integral for the force resultant (3.18) and written as the sum of two terms (3.14), (3.15) and (3.18):

$$\begin{aligned}
\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} &= \int_{-H/2}^{H/2} \begin{Bmatrix} \sigma_x(z) \\ \sigma_y(z) \\ \tau_{xy}(z) \end{Bmatrix} dz = \int_{-H/2}^{H/2} \left[\overline{Q}(z)\right] \begin{Bmatrix} \varepsilon_x(z) \\ \varepsilon_y(z) \\ \gamma_{xy}(z) \end{Bmatrix} dz \\
&= \int_{-H/2}^{H/2} \left[\overline{Q}(z)\right] \left(\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + z \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \right) dz \\
&= \left(\int_{-H/2}^{H/2} \left[\overline{Q}(z)\right] dz \right) \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + \left(\int_{-H/2}^{H/2} \left[\overline{Q}(z)\right] z dz \right) \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}
\end{aligned} \tag{3.20}$$

The first term in (3.20) captures the contribution of the mid-surface strains and the second term captures the contribution of the mid-surface curvatures to the force resultants. Equation (3.20) can be written as,

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = [A] \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + [B] \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \tag{3.21}$$

where the matrices $[A]$ and $[B]$ are defined as follows,

$$[A] = \int_{-H/2}^{H/2} \left[\overline{Q}\right] dz \quad , \quad [B] = \int_{-H/2}^{H/2} \left[\overline{Q}\right] z dz \tag{3.22}$$

Next, substituting for the stresses from (3.15) into the integral for the moment resultant (3.19) and integrating yields

$$
\begin{aligned}
\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} &= \int_{-H/2}^{H/2} \begin{Bmatrix} \sigma_x(z) \\ \sigma_y(z) \\ \tau_{xy}(z) \end{Bmatrix} z\,dz = \int_{-H/2}^{H/2} \left[\overline{Q}(z)\right] \begin{Bmatrix} \varepsilon_x(z) \\ \varepsilon_y(z) \\ \gamma_{xy}(z) \end{Bmatrix} z\,dz \\
&= \int_{-H/2}^{H/2} \left[\overline{Q}(z)\right] \left(\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + z \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \right) z\,dz \\
&= \left(\int_{-H/2}^{H/2} \left[\overline{Q}(z)\right] z\,dz \right) \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + \left(\int_{-H/2}^{H/2} \left[\overline{Q}(z)\right] z^2\,dz \right) \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}
\end{aligned}
\tag{3.23}
$$

where the first term captures the contribution of the mid-surface strains and the second term captures the contribution of the mid-surface curvatures to the moment resultants. Eqn. (3.23) can be written in the following form

$$
\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = [B] \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + [D] \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \tag{3.24}
$$

where matrix $[B]$ has been previously defined in (3.22) and matrix $[D]$ has the following definition

$$
[D] = \int_{-H/2}^{H/2} \left[\overline{Q}(z)\right] z^2 dz \tag{3.25}
$$

Eqns. (3.21) and (3.24) for the force and moment resultants can be combined into a single matrix equation as follows

$$
\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ \hline M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \left[\begin{array}{ccc|ccc} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ \hline B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{array} \right] \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \hline \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \tag{3.26}
$$

where the 6×6 matrix consisting of A_{ij}, B_{ij} and D_{ij} is known as the $[ABD]$ matrix. A_{ij} are the laminate extensional rigidities that relate the mid-surface strains to the force resultants. D_{ij} are the laminate bending/flexural rigidities that relate the curvatures to the moment resultants. B_{ij} are the laminate bending-extension coupling rigidities that relate the curvatures to the force resultants and the mid-surface strains to the moment resultants. Since the off-axis stiffness matrix $[\overline{Q}]$ is symmetric, the laminate stiffness matrices $[A]$, $[B]$ and $[D]$ are symmetric as well.

Eqn. (3.26) can be expressed compactly as

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \left[\begin{array}{c|c} A & B \\ \hline B & D \end{array}\right] \begin{Bmatrix} \varepsilon^0 \\ \kappa \end{Bmatrix} \tag{3.27}$$

where $\{N\}$, $\{M\}$, $\{\varepsilon^0\}$ and $\{\kappa\}$ are 3×1 column arrays of force resultants, moment resultants, mid-surface strains and mid-surface curvatures, respectively.

3.5.1 Computing the ABD matrices

The calculation of the laminate stiffness matrices $[A]$, $[B]$ and $[D]$ requires us to perform the through thickness integrations in Eqns. (3.22) and (3.25). Since the k^{th} layer extends from z_k to z_{k+1} and $[\overline{Q}(z)]$ is piecewise constant (i.e. constant in each layer or lamina of the laminate), the integrals can be transformed into a summation over the layers. The stiffness matrix $[A]$ is integrated as

$$\begin{aligned} [A] &= \int_{-H/2}^{H/2} \left[\overline{Q}\,(z)\right] dz = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \left[\overline{Q}\right]^{(k)} dz \\ &= \sum_{k=1}^{N} \left[\overline{Q}\right]^{(k)} \int_{z_k}^{z_{k+1}} dz \end{aligned} \tag{3.28}$$

Thus, the stiffness matrix $[A]$ can be calculated through the following summation,

$$[A] = \sum_{k=1}^{N} (z_{k+1} - z_k) \left[\overline{Q}\right]^{(k)} \tag{3.29}$$

Note that $z_{k+1} - z_k$ is the thickness h_k of lamina k. Therefore, the stiffness matrix $[A]$ can be determined through a layer by layer summation of the product of the lamina thickness h_k and off-axis stiffness matrix $[\overline{Q}]^{(k)}$.

The laminate stiffness matrices $[B]$ and $[D]$ can be similarly obtained through a summation over all the layers

$$[B] = \frac{1}{2} \sum_{k=1}^{N} \left(z_{k+1}^2 - z_k^2 \right) \left[\overline{Q} \right]^{(k)}$$
$$[D] = \frac{1}{3} \sum_{k=1}^{N} \left(z_{k+1}^3 - z_k^3 \right) \left[\overline{Q} \right]^{(k)} \tag{3.30}$$

The individual terms of the $[A]$, $[B]$ and $[D]$ matrices can be evaluated separately if needed. For example,

$$B_{ij} = \frac{1}{2} \sum_{k=1}^{N} \left(z_{k+1}^2 - z_k^2 \right) \overline{Q}_{ij}^{(k)} \tag{3.31}$$

3.5.2 Inversion of the load-deformation relations

If the force and moment resultants are know, the mid-surface strains and curvatures can be obtained by inverting (3.27).

$$\begin{Bmatrix} \varepsilon^0 \\ \kappa \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix}^{-1} \begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} a & b \\ b^T & d \end{bmatrix} \begin{Bmatrix} N \\ M \end{Bmatrix} \tag{3.32}$$

where the 6×6 $[abd]$ matrix is the inverse of the 6×6 $[ABD]$ matrix, i.e.,

$$\begin{bmatrix} a & b \\ b^T & d \end{bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix}^{-1} \tag{3.33}$$

The $[abd]$ matrix is a 6×6 matrix of laminates compliances. The matrices $[a]$ and $[d]$ are symmetric. However, the matrix $[b]$ need not be symmetric.

3.5.3 Elastic Couplings

The first three rows of (3.32) relate the mid-surface strains to the force and moment resultants

$$\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{16} \\ a_{12} & a_{22} & a_{26} \\ a_{16} & a_{26} & a_{66} \end{bmatrix} \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{16} \\ b_{21} & b_{22} & b_{26} \\ b_{61} & b_{62} & b_{66} \end{bmatrix} \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} \tag{3.34}$$

while last three rows of (3.32) relate the mid-surface curvatures to the force and moment resultants

$$\begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} = \begin{bmatrix} b_{11} & b_{21} & b_{61} \\ b_{12} & b_{22} & b_{62} \\ b_{16} & b_{26} & b_{66} \end{bmatrix} \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} + \begin{bmatrix} d_{11} & d_{12} & d_{16} \\ d_{12} & d_{22} & d_{26} \\ d_{16} & d_{26} & d_{66} \end{bmatrix} \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} \tag{3.35}$$

Here a_{ij} are the laminate extensional compliances that relate the force resultants to the mid-surface strains. d_{ij} are the laminate bending/flexural compliances that relate the moment resultants to the mid-surface curvatures. b_{ij} are the laminate bending-extension coupling compliances that relate the moment resultants to the mid-surface strains and the force resultants to the curvatures. The laminate compliances capture the response of a laminated composite material to applied loads. Laminated composite plates can exhibit behaviors that are not seen in isotropy plates.

In plane shear-extension coupling
The compliances a_{16} and a_{26} capture the influence of the in-plane axial forces on the in-plane shear strains and the in-plane shear forces on the in-plane normal strains. For example, if $a_{16} \neq 0$, an axial load N_x will induce a shear strain γ^0_{xy}. Similarly, a shear load N_{xy} will induce a normal strain ε^0_x.

Bending-twisting coupling
These compliances d_{16} and d_{26} capture the influence of the bending moments on the twisting curvatures and the twisting moment on the bending curvatures. For example, if $d_{16} \neq 0$, a bending moment M_x will induce a twisting curvature κ_{xy} and a twisting moment M_{xy} will induce a bending curvature κ_x.

Bending-extension coupling
The compliances b_{ij} couple the moment resultants to the mid-surface strains and the force resultants to the curvatures. For example, if $b_{11} \neq 0$, N_x will induce a curvature κ_x and M_x will induce an in-plane strain ε^0_x.

3.6 Laminate nomenclature and special types of laminates

3.6.1 Stacking sequence

In this section, we introduce the notation for specifying the fiber orientations of a laminate, known as the stacking sequence. The stacking sequence is specified in the form of an array of values enclosed in square brackets that contains the fiber orientations (in degrees) of the individual layers separated by the slash (/) symbol starting with the bottom layer and ending in the top layer, i.e., $[\theta_1/\theta_2/\cdots/\theta_k/\cdots/\theta_N]$. Fig. 3.9 shows a representative 5-layer laminate that has stacking sequence of [0/45/90/−45/0].

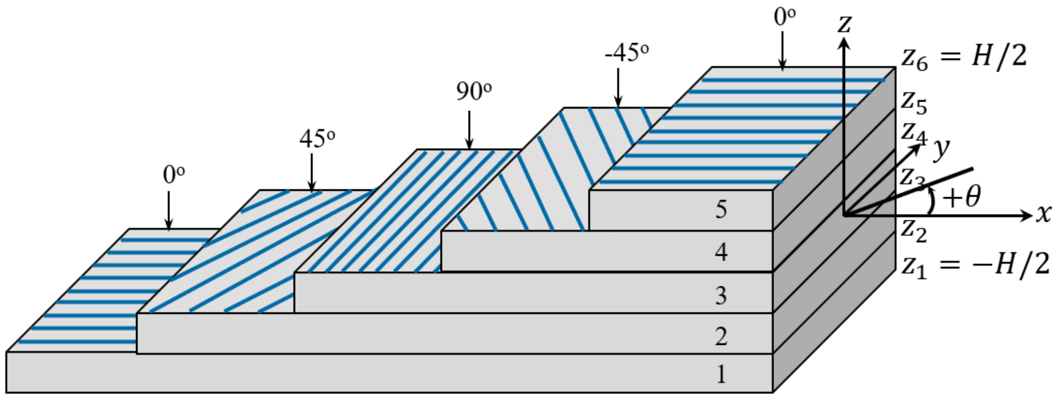


Figure 3.9: Representative [0/45/90/-45/0] ply stacking sequence

We use the following convention when specifying the stacking sequence.

1. In the case of laminates that are symmetric about the mid-surface, known as *symmetric laminates*, the stacking sequence of the bottom half of the laminate is specified followed by a subscript S. For example, [45/-30/0/0/-30/45] is abbreviated as $[45/\text{-}30/0]_S$. Symmetric laminates with an odd number of layers are listed with a bar over the center layer to indicate that it straddles the mid-surface. For example,[0/45/90/45/0] is abbreviated as $[0/45/\overline{90}]_S$.

2. Adjacent layers with fiber orientations of $+\theta$ followed by $-\theta$ are abbreviated as $\pm\theta$. For exmaple, a stacking sequence of [0/30/±45] is equivalent to [0/30/45/-45]. Similarly, adjacent layers with fiber orientations $-\theta$ followed by $+\theta$ are abbreviated as $\mp\theta$, e.g. [0/30/∓45] is equivalent to [0/30/-45/45].

3. A subscript n is used to designate adjacent layers with the same fiber orientation. For example, [-30/90/90/45/0/-45] is abbreviated as $[\text{-}30/90_2/45/0/\text{-}45]$. Repeated groups of layers are listed in parenthesis with a subscript n, e.g., [45/-45/0/45/-45/0] is abbreviated as $[(\pm 45/0)_2]$.

3.6.2 Laminate stiffness for special types of laminates

Symmetric laminates
A laminate is said to be symmetric when for each layer on one side of the mid-surface, there is a corresponding layer on the other side with identical thickness, properties and orientation. In the case of symmetric laminates, the contributions of layers on opposite sides of the mid-surface to the laminate stiffnesses $[B]$ cancel out and we obtain

$$B_{ij} = 0 \;\text{ or }\; [B] = [0]$$

Balanced laminates
A laminate is said to be balanced if for every lamina whose fibers are oriented at a certain angle θ,

there is another lamina oriented at $-\theta$ somewhere in the laminate, e.g. [45/−30/0/−45/30]. In the case of balanced laminates,

$$A_{16} = A_{26} = 0$$

Cross-ply laminates

Cross-ply laminates have fibers oriented at either 0° or 90°, e.g., $[0/90]_S$. Since $\overline{Q}_{16} = \overline{Q}_{26} = 0$ for each layer, $A_{16} = A_{26} = 0$, $B_{16} = B_{26} = 0$ and $D_{16} = D_{26} = 0$.

Quasi-isotropic laminates

Quasi-isotropic laminates are special type of symmetric laminates with in-plane stiffness that behaves like that of an isotropic plate. That is,

$$A_{11} = A_{22} \quad , \quad A_{16} = A_{26} = 0 \quad , \quad A_{66} = \frac{A_{11} - A_{12}}{2}$$

Examples of quasi-isotropic laminates include $[0/90/45/-45]_S$, $[0/60/-60]_S$.

3.7 Laminate analysis procedure

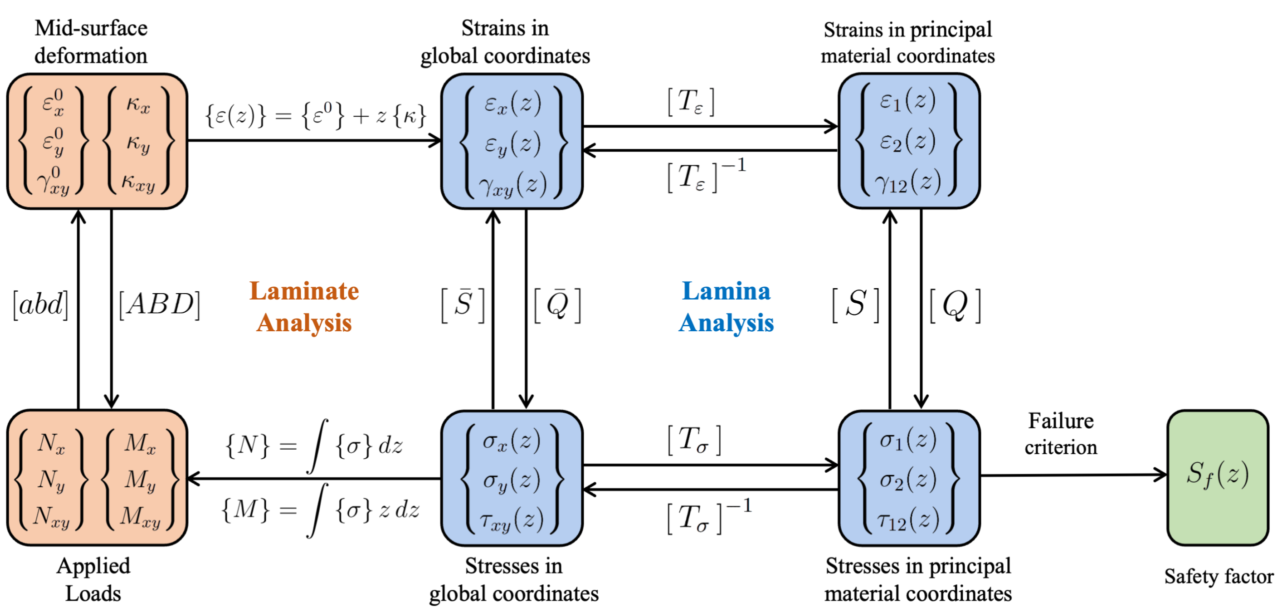


Figure 3.10: Concept map for the analysis of laminated composites. Adapted and modified from M.W. Hyer, Stress Analysis of Fiber-Reinforced Composite Materials, DEStech, 2009

3.8 Analysis of laminates subjected to in-plane loads

3.8.1 Thin laminated tubes subjected to an axial force and a torque

Consider a thin laminated composite tube with a symmetric ply layup. It is assumed that the mean radius R is much larger than the wall thickness H. The tube is subjected to an axial force P and a torque T, as shown in Fig. 3.11.

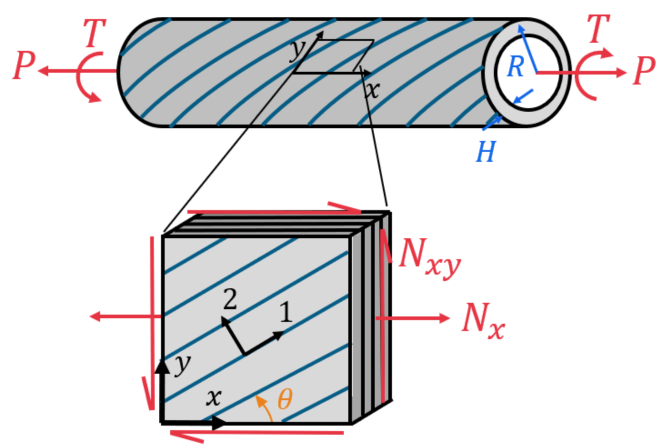


Figure 3.11: A thin-walled laminated tube subjected to axial and torsional loads

In the case of a thin-walled tube, we can treat an element as a flat laminate and relate the force and moment resultants to the applied loads as follows

$$N_x = \frac{P}{2\pi R} \quad , \quad N_{xy} = \frac{T}{2\pi R^2} \quad , \quad N_y = 0 \quad , \quad M_x = M_y = M_{xy} = 0 \tag{3.36}$$

The force and moment resultants can be substituted into (3.34) and (3.35) to obtain the mid-surface strains $\{\varepsilon^0\}$ and curvatures $\{\kappa\}$. Subsequently, we can calculate the stresses using (3.15) and analyze the safety factor using the Tsai-Wu failure theory.

3.8.2 Laminated composite pressure vessels

Consider a thin-walled pressure vessel with a mean radius R and wall thickness H that is subjected to an internal pressure p as shown in Fig. 3.12. The pressure vessel is assumed to have a symmetric layup.

The in-plane force resultants in the axial and hoop direction are obtained using static equilibrium.

$$N_x = \frac{pR}{2} \quad , \quad N_y = pR \quad , \quad N_{xy} = 0 \tag{3.37}$$

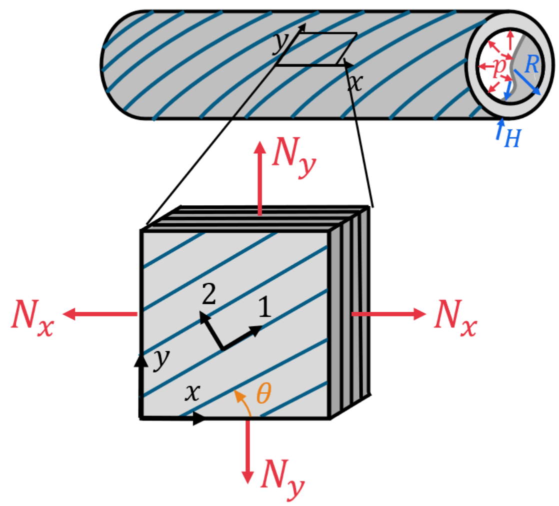


Figure 3.12: A thin-walled laminated pressure vessel with rigid end caps, subjected to an internal pressure

The moment resultants are assumed to be zero.

$$M_x = M_y = M_{xy} = 0 \tag{3.38}$$

3.9 Analysis of laminated composite beams

Consider a laminated beam of length L and width W. The thickness H of the laminated beam is assumed to be much smaller than the length L. The beam is assumed to have a symmetric layup to preclude bending-extension coupling effects.

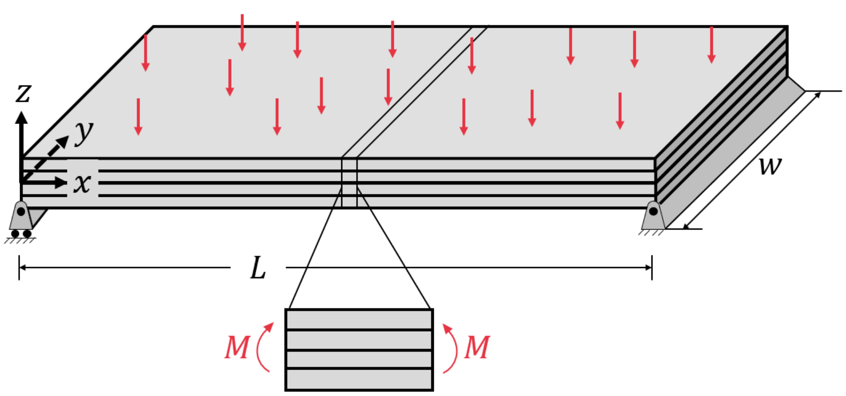


Figure 3.13: A thin laminated beam in bending

3.9.1 Stress analysis of laminated beams

As we would for a regular beam, we can draw the bending moment diagram and use it to determine the bending moment M at the spanwise location of interest. The moment resultant M_x is the bending moment per unit width. That is,

$$M_x = -\frac{M}{W} \tag{3.39}$$

The negative sign is introduce to account for the fact that the positive convention for the bending moment M is opposite to that of the positive direction for M_x. All other force and moment resultants are zero, i.e., $N_x = N_y = N_{xy} = 0, M_y = M_{xy} = 0$. Next, we can evaluate the mid-surface strains and curvatures and obtain the through-thickness variation of stresses.

3.9.2 Deflection of laminated beams

In the case of symmetric beam, the laminate compliance matrix $[b] = 0$ and (3.35) simplifies to

$$\begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} = \begin{bmatrix} d_{11} & d_{12} & d_{16} \\ d_{12} & d_{22} & d_{26} \\ d_{16} & d_{26} & d_{66} \end{bmatrix} \begin{Bmatrix} M_x \\ \cancelto{0}{M_y} \\ \cancelto{0}{M_{xy}} \end{Bmatrix} \tag{3.40}$$

Therefore, the curvature κ_x in the spanwise direction,

$$\kappa_x = d_{11} M_x = -\frac{d_{11} M}{W} \tag{3.41}$$

from which we can obtain the moment-curvature relationship

$$M = \underbrace{\left(\frac{W}{d_{11}}\right)}_{\text{bending rigidity}} \cdot \underbrace{(-\kappa_x)}_{\text{beam curvature}} \tag{3.42}$$

where W/d_{11} is the rigidity of the laminated beam. Therefore, we can replace the laminate beam with an equivalent homogeneous beam with effective flexural modulus $\overline{E}_x^{fl}$ that is obtained by equating the equivalent bending rigidity $\overline{E}_x^{fl} I$ to the effective rigidity W/d_{11} of the laminate where $I = WH^3/12$ is the moment of inertia of the equivlaent homgoeneous beam

$$\overline{E}_x^{fl} I = \frac{W}{d_{11}} \quad \Rightarrow \quad \overline{E}_x^{fl}\left(\frac{1}{12} WH^3\right) = \frac{W}{d_{11}} \tag{3.43}$$

from which we obtain the effective flexural modulus $\overline{E}_x^{fl}$

$$\overline{E}_x^{fl} = \frac{12}{d_{11}H^3} \tag{3.44}$$

Thus, the deflection of a laminated composite beam can be obtained by replacing the laminated beam with a homogeneous beam of flexural modulus $\overline{E}_x^{fl}$ and using the strength of materials expressions for deflection.

EXAMPLE 3.1: Deflection of a laminated cantilever beam

Consider a laminated cantilever beam of length L that is subjected to a concentrated force P at the tip.

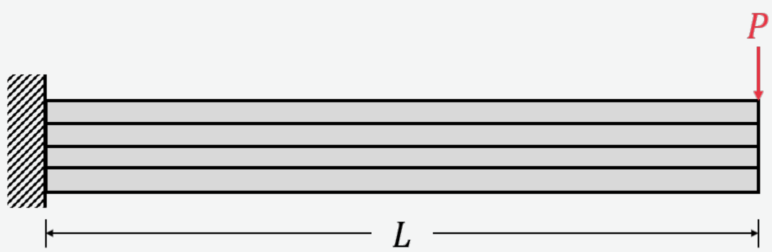


Figure 3.14: Laminated cantilever beam

The tip deflection δ is obtained using the beam theory formula by replacing the Young's modulus E by the flexural modulus $\overline{E}_x^{fl}$

$$\delta = \frac{PL^3}{3\overline{E}_x^{fl} I} \tag{3.45}$$

where $\overline{E}_x^{fl}$ can be determined from the bending compliance d_{11} and beam thickness H using (3.44)

Exercises

Use the representative properties in Sec. 1.10.1 for unidirectional carbon/epoxy composites and the representative properties in Sec. 1.10.2 for fabric-reinforced carbon/epoxy composites. Assume that the ply thickness h of the unidirectional and fabric-reinforced laminae are 0.2 mm and 0.4 mm, respectively. Use the Tsai-Wu theory for failure analysis.

3.1 Consider a [0/45] unidirectional carbon fiber-reinforced composite laminate. The laminate experiences the following mid-surface strains and curvatures when subjected to certain loads: $\varepsilon_x^0 = 1000\,\mu$, $\kappa_x = -10\text{ m}^{-1}$ and $\kappa_{xy} = 10\text{ m}^{-1}$. All the other mid-surface strains and curvatures are zero.

(a) Plot the through-thickness variation of the strain $\varepsilon_x, \varepsilon_y$ and γ_{xy} (in $\mu\varepsilon$)

(b) Determine the stresses σ_x, σ_y and τ_{xy} at the top and bottom surfaces of each lamina in the global coordinate system and plot the through-thickness variation of the stresses σ_x, σ_y and τ_{xy} (in MPa)

(c) Determine the stresses σ_1, σ_2 and τ_{12} at the top surface of the laminate in the material coordinate system.

3.2 Consider a symmetric balanced laminate with known laminate stiffnesses A_{ij}, compliances a_{ij} and thickness H.

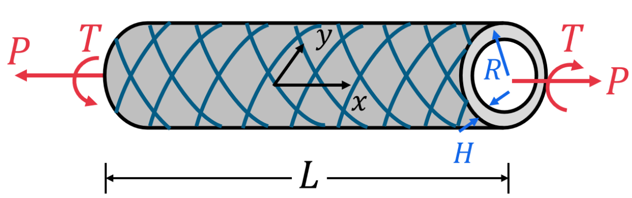
(a) Derive expressions for the effective in-plane laminate elastic moduli $\bar{E}_x, \bar{E}_y, \bar{\nu}_{xy}$ and $\bar{G}_{xy}$ in the x-y global coordinate system in terms of A_{ij} or a_{ij} and thickness H.

(b) Obtain the effective elastic moduli $\bar{E}_x, \bar{E}_y, \bar{\nu}_{xy}$ and $\bar{G}_{xy}$ of a [0/90/0] unidirectional carbon fiber-reinforced composite in the x-y global coordinate system. Is $\bar{E}_x$ or $\bar{E}_y$ larger and why? How does $\bar{G}_{xy}$ compare with the shear modulus G_{12} of a lamina? Discuss your results.

(c) Consider a $[0/90/\pm45]_S$ quasi-isotropic laminate made of unidirectional carbon fiber-reinforced plies. Calculate the elastic moduli $\bar{E}_x, \bar{E}_y, \bar{\nu}_{xy}$ and $\bar{G}_{xy}$ in the x-y global coordinate system. Does the laminate behave like an isotropic plate, i.e., is $\bar{E}_x = \bar{E}_y$ and $G_{xy} = E_x/2(1+\nu_{xy})$? Are the bending rigidities D_{11} and D_{22} the same as is the case for isotropic plates?

(d) If you rotate the entire quasi-isotropic laminate considered in part (c) by an arbitrary angle α, say 30°, do the elastic moduli $\bar{E}_x, \bar{E}_y, \bar{\nu}_{xy}$ and $\bar{G}_{xy}$ change? Do the bending rigidities D_{11} and D_{22} change after the laminate is rotated? You can check your answer by rotating all the plies by an angle of 30°.

3.3 Consider a $[\pm30]_S$ thin-walled laminated tube made of *woven fabric-reinforced* plies. The tube has a mean radius $R = 5$ cm and length $L = 0.5$ m. It is subjected to a torque of $T = 2$ kN·m and an axial force of $P = 20$ kN. The x and y- directions are oriented parallel to the axial and circumferential directions, respectively.



(a) Determine the midsurface strains and curvatures. Discuss their relative values and whether they make sense.

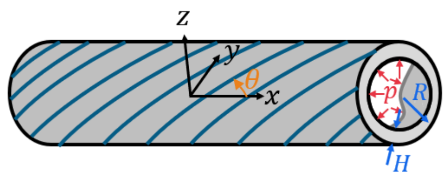
(b) Determine the overall elongation (in mm) of the tube using the midsurface strain ε_x^0 and length L

(c) Derive an expression for the total twist of the tube (rotation of one end relative to the other) using the mid-surface shear strain γ_{xy}^0, the length L and the radius R. Use the obtained expression to calculate the total twist in degrees.

(d) Plot the through-thickness variation of the strain ε_x and stress components σ_x, τ_{xy} and τ_{12}. Which layer(s) exhibit the largest shear stress τ_{12}?

(e) Plot the through-thickness variation of the factor of safety S_{fa}. Determine the minimum factor of safety S_{fa}^{min} and the corresponding layer(s).

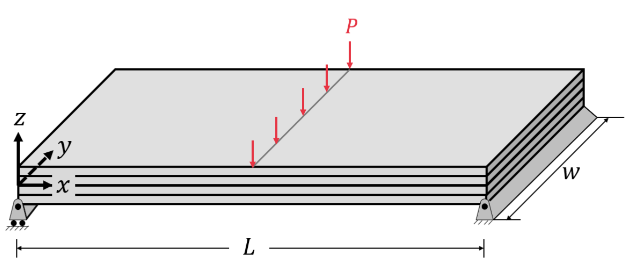
3.4 Consider a cross-ply (i.e., 0° or 90° plies) laminated composite pressure vessel made of unidirectional carbon fiber-reinforced laminae. It has a mean radius of $R = 0.25$ m and is subjected to an internal gauge pressure of $p = 1.25$ MPa.



(a) Consider the case where the pressure vessel is made of a [0/90/0] cross-ply laminate. Calculate the transverse normal stress σ_2 in the plies and compare it with the transverse tensile strength F_{2t}. Determine the factor of safety S_{fa}. Will the pressure vessel be able to withstand the internal pressure?

(b) Is there another *symmetric*, *cross-ply* lamination scheme that will give a higher factor of safety than the one in part (a), preferably without increasing the weight? If so, specify the stacking sequence, the corresponding safety factor and the reason why it has a higher factor of safety.

3.5 Consider a symmetric cross-ply laminated simply supported beam of solid rectangular cross-section made of unidirectional carbon fiber-reinforced laminae. The beam is of length $L = 20$ cm and width $W = 5$ cm. It is subjected to a net force of $P = 5$ N at the mid-span as shown in the figure.



(a) If the beam has a $[90/0]_S$ stacking sequence, plot the through-thickness variation of the stress components σ_x, σ_2 and the factor of safety S_{fa} at the location where the bending moment is largest. Determine the minimum factor of safety S_{fa}^{min}. Where is the factor of safety the lowest (specify the layer and through-thickness location)? Can you explain why the factor of safety is lowest at that location?

(b) Determine the mid-span deflection δ of the $[90/0]_S$ laminated beam considered in part (a).

(c) If the beam has a $[0/90]_S$ stacking sequence, determine the minimum factor of safety S_{fa}^{min} and compare it with the value obtained for a $[90/0]_S$ in part (a). Where is the factor of safety the lowest? Can you explain why the factor of safety is lowest at that location?

(d) Compare the deflection of the $[0/90]_S$ beam considered in part (c) with the value obtained in part (b) for a $[90/0]_S$ beam. Are the deflections significantly different? If so, can you explain why?

Equations of Motion for a Plate 4

In this chapter, we will derive the equations of motion of a laminated composite plate in terms of the force and moment resultants from the three dimensional equations of motion.

4.1 Analysis of laminated composite plates

Consider a laminated rectangular plate of length a in the x-direction and width b in the y-direction as shown in Fig. 4.1. The laminated plate is composed of N laminae and has a total thickness H. It is subjected to a distributed load of magnitude $q(x, y, t)$ in the positive z-direction. The distributed load can act either on the top surface ($z = H/2$) or the bottom surface ($z = -H/2$). The distributed load can have a non-uniform spatial variation with respect to the in-plane coordinates x and y. In addition, the load can vary with time t in the case of dynamic loading (e.g., forced vibration).

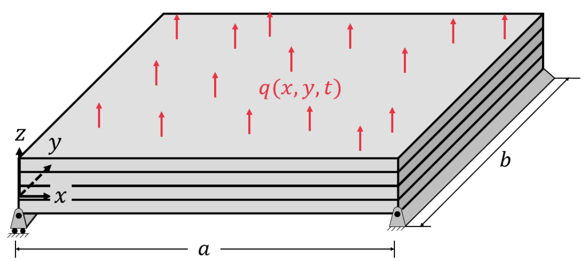


Figure 4.1: Rectangular laminated plate subjected to loads

In general, the force and moment resultants acting on element of a laminated composite plate vary from point to point. When analyzing laminated plates, we use the equations for motion and boundary conditions to determine the force resultants, the moment resultants, mid-surface strains and curvatures at a location.

4.1.1 Transverse Shear Force Resultants

In addition to the in-plane force resultants N_x, N_y and N_{xy}, and the moment resultants M_x, M_y and M_{xy} defined earlier in Sec. 3.4, we define the transverse shear force resultants V_x and V_y as follows,

$$V_x = \int_{-H/2}^{H/2} \tau_{xz} dz , \qquad V_y = \int_{-H/2}^{H/2} \tau_{yz} dz \tag{4.1}$$

The transverse shear force resultants V_x and V_y are the vertical forces per unit width acting on the surfaces normal to the x and y-axes, respectively, as shown in Fig. 4.2.

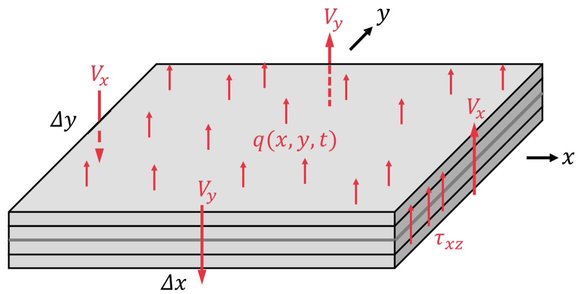


Figure 4.2: Shear force resultants acting on an element

4.2 Equations of motion for a laminated plate

4.2.1 Three-dimensional equations of motion

To derive the equations of motion of a laminated plate in terms of the force and moment resultants, we start with the three dimensional equations of motion for an elastic body, namely [1]

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = \rho \frac{\partial^2 u}{\partial t^2} \tag{4.2a}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = \rho \frac{\partial^2 v}{\partial t^2} \tag{4.2b}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = \rho \frac{\partial^2 w}{\partial t^2} \tag{4.2c}$$

where ρ is the mass density of the material. Note that we have neglected body forces for the sake of simplicity.

4.2.2 Equations of motion in terms of force and moment resultants

We can integrate the equations of motion through the thickness of a laminated plate to obtain the equations of motion in terms of the force and moment resultants. Since the equations of motion are satisfied at every point within a laminated plate, we can integrate them through the thickness of the laminated plate and the resulting equations should also be satisfied.

(a) Equation of motion in the x-direction

We integrate Eqn. (4.2a) through the thickness (i.e. with respect to z)

$$\int_{-H/2}^{H/2} \left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} \right) dz = \int_{-H/2}^{H/2} \rho \frac{\partial^2}{\partial t^2} \left(u_0 - z \frac{\partial w_0}{\partial x} \right) dz \tag{4.3}$$

$$\frac{\partial}{\partial x} \int_{-H/2}^{H/2} \sigma_x dz + \frac{\partial}{\partial y} \int_{-H/2}^{H/2} \tau_{xy} dz + \tau_{xz} \Big|_{-H/2}^{H/2} = \frac{\partial^2 u_0}{\partial t^2} \int_{-H/2}^{H/2} \rho dz - \frac{\partial^2}{\partial t^2} \left(\frac{\partial w_0}{\partial x} \right) \int_{-H/2}^{H/2} \rho z dz \tag{4.4}$$

Next, based on the definition of in-plane force resultants, we obtain

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} + \cancelto{0}{\tau_{xz} (H/2)} - \cancelto{0}{\tau_{xz} (-H/2)} = I_0 \frac{\partial^2 u_0}{\partial t^2} - I_1 \frac{\partial^2}{\partial t^2} \left(\frac{\partial w_0}{\partial x} \right) \tag{4.5}$$

which reduces to the following equation of motion in terms of force resultants since there are no shear stresses acting on the top and bottom surfaces of the plate, i.e., $\tau_{xz} (-H/2) = \tau_{xz} (H/2) = 0$,

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = I_0 \frac{\partial^2 u_0}{\partial t^2} - I_1 \frac{\partial^2}{\partial t^2} \left(\frac{\partial w_0}{\partial x} \right) \tag{4.6}$$

where I_i are integrals that involve the density and are defined as

$$I_i = \int_{-H/2}^{H/2} \rho (z) z^i dz \tag{4.7}$$

The integrals I_i can be written in array form for $i = 0, 1, 2$ as

$$\begin{Bmatrix} I_0 \\ I_1 \\ I_2 \end{Bmatrix} = \int_{-H/2}^{H/2} \rho (z) \begin{Bmatrix} 1 \\ z \\ z^2 \end{Bmatrix} dz \tag{4.8}$$

Physically, I_0 is the mass of the plate per unit area (i.e. areal mass) and I_2 is known as "rotary inertia". Since the density is usually constant for each layer, the integrals (4.8) can be evaluated through a layer by layer integration and expressed as

$$I_i = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \rho_k z^i dz = \frac{1}{(i+1)} \sum_{k=1}^{N} \rho_k \left[z_{k+1}^{(i+1)} - z_k^{(i+1)} \right] \tag{4.9}$$

We thus obtain the following expressions for I_0, I_1 and I_2

$$I_0 = \sum_{k=1}^{N} \rho_k \left(z_{k+1} - z_k\right) = \sum_{k=1}^{N} \rho_k h_k, \quad I_1 = \frac{1}{2} \sum_{k=1}^{N} \rho_k \left(z_{k+1}^2 - z_k^2\right), \quad I_2 = \frac{1}{3} \sum_{k=1}^{N} \rho_k \left(z_{k+1}^3 - z_k^3\right) \tag{4.10}$$

The expressions for I_0, I_1 and I_2 in (4.10) are valid for a hybrid laminate where each ply may have a different density. In the case of laminate where all the laminae have the same density ρ, the integral (4.7) for I_i reduces to

$$I_i = \rho \int_{-H/2}^{H/2} z^i dz = \frac{\rho}{(i+1)} z^{(i+1)} \Big|_{-H/2}^{H/2} \tag{4.11}$$

and we obtain

$$I_0 = \rho H, \qquad I_1 = 0, \qquad I_2 = \frac{\rho H^3}{12} \qquad \text{if} \quad \rho_k = \rho \tag{4.12}$$

(b) Equation of motion in the y-direction

We Integrate (4.2b) through the thickness (i.e. with respect to z)

$$\int_{-H/2}^{H/2} \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} \right) dz = \int_{-H/2}^{H/2} \rho \frac{\partial^2}{\partial t^2} \left(v_0 - z \frac{\partial w_0}{\partial y} \right) dz \tag{4.13}$$

Integrating term by term and using the definition of force resultants gives

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} + \cancelto{0}{\tau_{yz}(H/2)} - \cancelto{0}{\tau_{yz}(-H/2)} = I_0 \frac{\partial^2 v_0}{\partial t^2} - I_1 \frac{\partial^2}{\partial t^2} \left(\frac{\partial w_0}{\partial y} \right) \tag{4.14}$$

Since there are no shear stresses on the top and bottom surfaces, i.e. $\tau_{yz}(-H/2) = \tau_{yz}(H/2) = 0$, we obtain

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = I_0 \frac{\partial^2 v_0}{\partial t^2} - I_1 \frac{\partial^2}{\partial t^2} \left(\frac{\partial w_0}{\partial y} \right) \tag{4.15}$$

(c) Equation of motion in the z-direction

We integrate (4.2c) through the thickness (i.e. with respect to z)

$$\int_{-H/2}^{H/2} \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} \right) dz = \int_{-H/2}^{H/2} \rho \frac{\partial^2 w_0}{\partial t^2} dz \tag{4.16}$$

Integrating term by term and using the definition of the shear force resultants in (4.1), gives

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \sigma_z\left(H/2\right) - \sigma_z\left(-H/2\right) = I_0 \frac{\partial^2 w_0}{\partial t^2} \tag{4.17}$$

The distributed load $q(x, y, t)$ can act on either the top or bottom surface, as shown in Fig. 4.3.

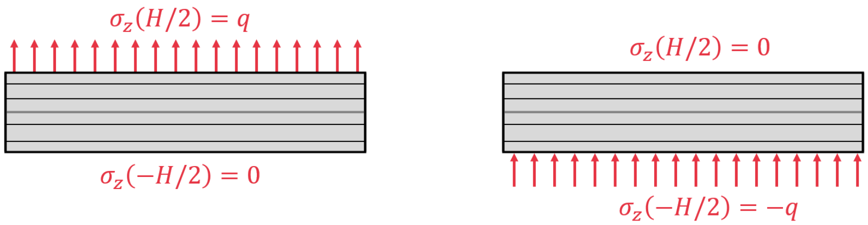


Figure 4.3: Distributed load acting on the top and bottom surfaces of a laminate

If the distributed load is applied on the top surface then, $\sigma_z\left(H/2\right) = q\left(x, y, t\right)$ and $\sigma_z\left(-H/2\right) = 0$. On the other hand, if the distributed load is applied on the bottom surface, then $\sigma_z\left(-H/2\right) = -q\left(x, y, t\right)$ and $\sigma_z\left(H/2\right) = 0$. In both cases, we obtain

$$\sigma_z\left(H/2\right) - \sigma_z\left(-H/2\right) = q\left(x, y, t\right) \tag{4.18}$$

Substituting from (4.18) into (4.17) gives

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + q\left(x, y, t\right) = I_0 \frac{\partial^2 w_0}{\partial t^2} \tag{4.19}$$

(d) First moment of the equation of motion in the x-direction

We multiply (4.2a) by z and integrate with respect to z

$$\int_{-H/2}^{H/2} \frac{\partial \sigma_x}{\partial x} z dz + \int_{-H/2}^{H/2} \frac{\partial \tau_{xy}}{\partial y} z dz + \int_{-H/2}^{H/2} \frac{\partial \tau_{xz}}{\partial z} z dz = \int_{-H/2}^{H/2} \rho \left(\frac{\partial^2 u_0}{\partial t^2} - z \frac{\partial^2}{\partial t^2}\left(\frac{\partial w_0}{\partial x}\right)\right) z dz \tag{4.20}$$

from which it follows that,

$$\frac{\partial}{\partial x}\int_{-H/2}^{H/2} \sigma_x z dz + \frac{\partial}{\partial y}\int_{-H/2}^{H/2} \tau_{xy} z dz + \int_{-H/2}^{H/2}\left[\frac{\partial}{\partial z}\left(z\tau_{xz}\right) - \tau_{xz}\right] dz = I_1 \frac{\partial^2 u_0}{\partial t^2} - I_2 \frac{\partial^2}{\partial t^2}\left(\frac{\partial w_0}{\partial x}\right) \tag{4.21}$$

Evaluating the terms in equation (4.21) gives

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} + \cancelto{0}{z\tau_{xz}\Big|_{-H/2}^{H/2}} - V_x = I_1 \frac{\partial^2 u_0}{\partial t^2} - I_2 \frac{\partial^2}{\partial t^2}\left(\frac{\partial w_0}{\partial x}\right) \tag{4.22}$$

Since the shear stress $\tau_{xz} = 0$ on the top and bottom surface, (4.22) reduces to

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - V_x = I_1 \frac{\partial^2 u_0}{\partial t^2} - I_2 \frac{\partial^2}{\partial t^2}\left(\frac{\partial w_0}{\partial x}\right) \tag{4.23}$$

(e) First moment of the equation of motion in the y-direction

Using a process similar to the one in (d), we multiply (4.2b) by z and integrate with respect to z to obtain

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - V_y = I_1 \frac{\partial^2 v_0}{\partial t^2} - I_2 \frac{\partial^2}{\partial t^2}\left(\frac{\partial w_0}{\partial y}\right) \tag{4.24}$$

4.2.3 Equations of motion for classical laminated plate theory

In the classical laminated plate theory, the governing equations are solved to obtain the three mid-surface displacements, namely $u_o(x, y, z, t)$, $v_o(x, y, z, t)$ and $w_o(x, y, z, t)$, from which the mid-surface strains, curvatures and through-the-thickness variation of stresses are obtained. The five equations of motion, namely (4.6), (4.15), (4.19), (4.23) and (4.24), are reduced to a system of three equations for u_o, v_o and w_o by eliminating V_x and V_y.

Differentiating equation (4.23) with respect to x and equation (4.24) with respect to y yields

$$\frac{\partial V_x}{\partial x} = \frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_{xy}}{\partial x \partial y} - I_1 \frac{\partial}{\partial x}\left(\frac{\partial^2 u_0}{\partial t^2}\right) + I_2 \frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 w_0}{\partial x^2}\right) \tag{4.25a}$$

$$\frac{\partial V_y}{\partial y} = \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} - I_1 \frac{\partial}{\partial y}\left(\frac{\partial^2 v_0}{\partial t^2}\right) + I_2 \frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 w_0}{\partial y^2}\right) \tag{4.25b}$$

Substituting equations (4.25a) and (4.25b) into equation (4.19) gives

$$\begin{aligned} &\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_{xy}}{\partial x \partial y} - I_1 \frac{\partial}{\partial x}\left(\frac{\partial^2 u_0}{\partial t^2}\right) + I_2 \frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 w_0}{\partial x^2}\right) + \frac{\partial^2 M_{xy}}{\partial x \partial y} + \\ &\frac{\partial^2 M_y}{\partial y^2} - I_1 \frac{\partial}{\partial y}\left(\frac{\partial^2 v_0}{\partial t^2}\right) + I_2 \frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 w_0}{\partial y^2}\right) + q\,(x, y, t) = I_0 \frac{\partial^2 w_0}{\partial t^2} \end{aligned} \tag{4.26}$$

from which it follows that

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + q\,(x, y, t) = I_0 \frac{\partial^2 w_0}{\partial t^2} + I_1 \frac{\partial^2}{\partial t^2}\left(\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y}\right) - I_2 \frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial^2 w_0}{\partial y^2}\right) \tag{4.27}$$

Equations (4.6), (4.15) and (4.27) are the equations of motion for a laminated plate.

4.2.4 Equilibrium equations for classical laminated plate theory

In the case of static loading, the displacements do not vary with time. Hence the partial derivatives with respect to time on the right hand side of (4.6), (4.15) and (4.27) are zero and we obtain the following equilibrium equations for a laminated plate

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \tag{4.28a}$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0 \tag{4.28b}$$

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + q\,(x, y) = 0 \tag{4.28c}$$

Upon solving the boundary value problem and obtaining the bending moments M_x, M_y and the twisting moment M_{xy}, we can obtain the transverse shear force resultants from (4.23) and (4.24) as follows

$$V_x = \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y}\,, \quad V_y = \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} \tag{4.29}$$

4.3 Physical interpretation of the equilibrium equations

The equilibrium equations (4.28) can be obtained by considering the equilibrium of forces and moments acting on element.

4.3.1 Force balance in the x-direction in terms of resultants

Since the force and moment resultants vary from point to point in a laminated plate, we can use a Taylor series expansion to represent the resultants acting on the edges in terms of the resultants at the center of an element. Thus the axial force resultant N_x on edge BC that is at a distance of $\Delta x/2$ from the center is $N_x + \frac{\partial N_x}{\partial x}\frac{\Delta x}{2}$ as shown in Fig. 4.4. To obtain the axial force acting on the edge BC, the force resultant needs to be multiplied by the width Δy. Thus the axial force in the x-direction on edge BC is $(N_x + \frac{\partial N_x}{\partial x}\frac{\Delta x}{2})\Delta y$. Similarly, we can express the axial forces in the x-direction on the the other three

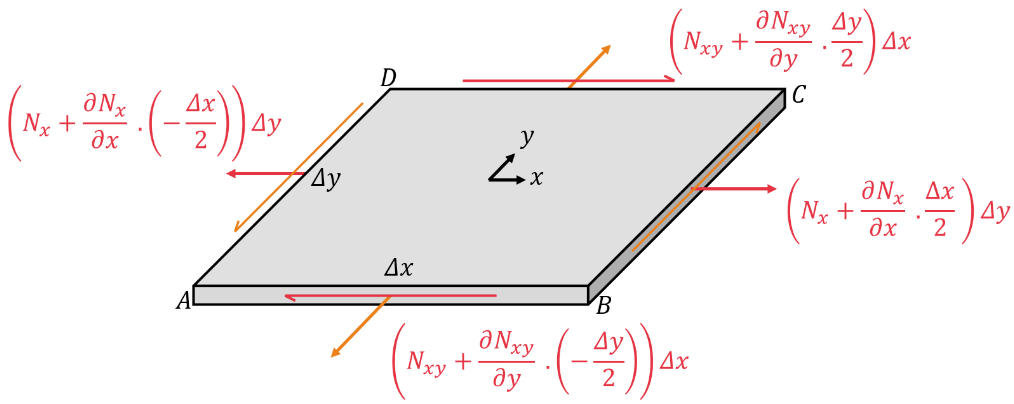


Figure 4.4: Forces acting on an element in the x-direction

edges. The equilibrium equation in the x-direction is obtained by summing the forces in the x direction,

$$\left(\cancel{N_x} + \frac{\partial N_x}{\partial x}\frac{\Delta x}{2}\right)\Delta y + \left(\cancel{N_{xy}} + \frac{\partial N_{xy}}{\partial y}\frac{\Delta y}{2}\right)\Delta x - \left(\cancel{N_x} - \frac{\partial N_x}{\partial x}\frac{\Delta x}{2}\right)\Delta y - \left(\cancel{N_{xy}} - \frac{\partial N_{xy}}{\partial y}\frac{\Delta y}{2}\right)\Delta x = 0 \quad (4.30)$$

After canceling terms and dividing through by $\Delta x \cdot \Delta y$ we obtain

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \quad (4.31)$$

This is identical to the equilibrium equation (4.28a) obtained earlier. Similarly, the following equilibrium equation (4.28b) can be obtained by summing the forces in the y direction.

It is possible to obtain the equilibrium equation in the z-direction by considering the vertical forces acting on an element as shown in Fig.4.5. Summing forces in the z direction gives

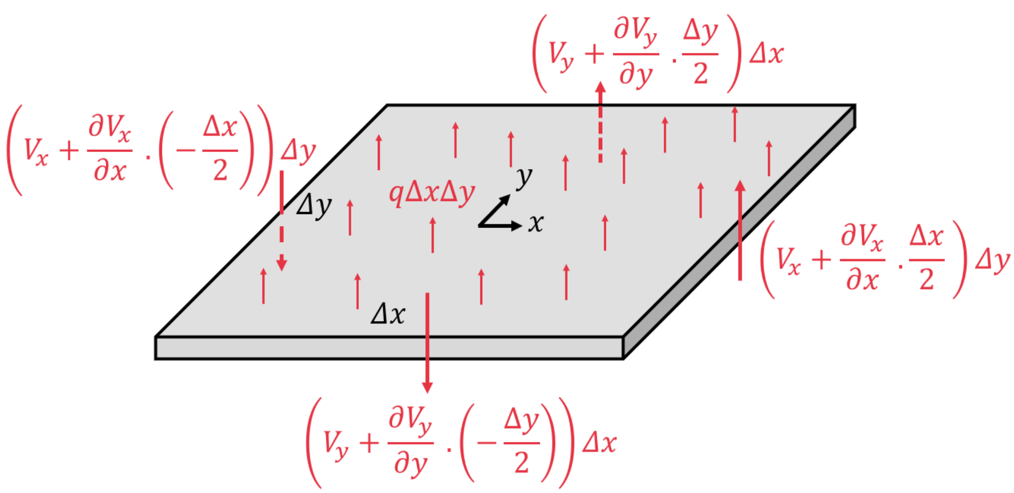


Figure 4.5: Forces acting on an element in the z-direction

$$\left(\cancel{V_x} + \frac{\partial V_x}{\partial x}\frac{\Delta x}{2}\right)\Delta y + \left(\cancel{V_y} + \frac{\partial V_y}{\partial y}\frac{\Delta y}{2}\right)\Delta x - \left(\cancel{V_x} - \frac{\partial V_x}{\partial x}\frac{\Delta x}{2}\right)\Delta y - \left(\cancel{V_y} - \frac{\partial V_y}{\partial y}\frac{\Delta y}{2}\right)\Delta x + q\,(x, y)\,\Delta x\Delta y = 0 \tag{4.32}$$

After canceling terms and dividing through by $\Delta x \cdot \Delta y$ we obtain

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + q\,(x, y) = 0 \tag{4.33}$$

which is identical to Eqn. (4.19) for quasi-static loading. The other equilibrium equations are obtained by summing the moments about the x and the y axes.

4.4 Boundary conditions

When solving boundary value problems, we need to define appropriate boundary conditions that model realistic support conditions.

4.4.1 Clamped Boundaries

Consider a rectangular laminated plate that is clamped on all four edges as shown in Fig. 4.6

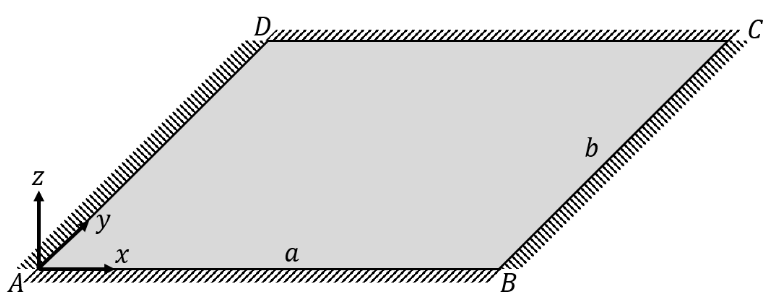


Figure 4.6: Clamped rectangular plate

The boundary conditions along edges AD and BC are

$$w_0 = 0\,,\ \frac{\partial w_0}{\partial x} = 0\,,\ u_0 = 0\,,\ v_0 = 0 \quad \text{at} \quad x = 0, a \tag{4.34}$$

and along edges AB and DC are

$$w_0 = 0\,,\ \frac{\partial w_0}{\partial y} = 0\,,\ u_0 = 0\,,\ v_0 = 0 \quad \text{at} \quad y = 0, b \tag{4.35}$$

It is possible to have other "clamped" support conditions. For example, consider a laminated plate that is supported as shown in Fig. 4.7. In this case, the force resultants N_x and N_{xy} are negligible at $x = a$

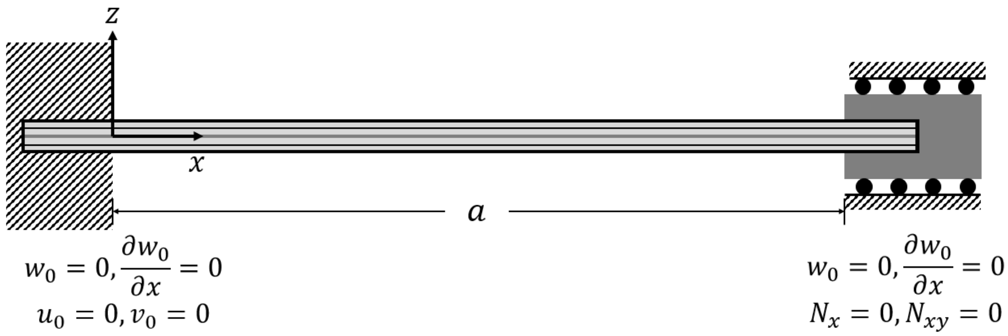


Figure 4.7: Different types of clamped boundary conditions

due to the roller supports and hence the boundary conditions can be specified as

$$w_0 = 0\,,\ \frac{\partial w_0}{\partial x} = 0\,,\ N_x = 0\,,\ N_{xy} = 0 \quad \text{at} \quad x = a \tag{4.36}$$

4.4.2 Simply Supported Boundary Conditions

It is possible to have four different types of simply supported boundary conditions on the edges $x = 0$ and $x = a$. They are commonly classified as S_1, S_2, S_3 and S_4 simply supported boundary conditions [3, 4].

$\mathbf{S_1}$ **boundary condition**: Axially fixed, transversely fixed

$$w_0 = 0\,,\ M_x = 0\,,\ u_0 = 0\,,\ v_0 = 0 \tag{4.37}$$

$\mathbf{S_2}$ **boundary condition**: Axially free, transversely fixed

$$w_0 = 0\,,\ M_x = 0\,,\ N_x = 0\,,\ v_0 = 0 \tag{4.38}$$

S_1 and S_2 boundary conditions are illustrated in Fig. 4.8. The other two simply supported boundary

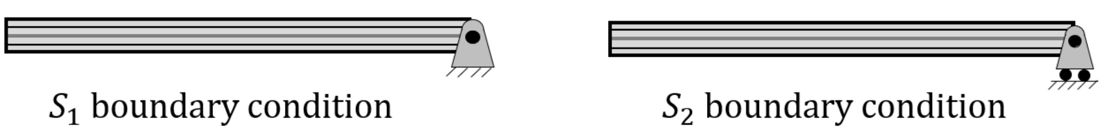


Figure 4.8: Different simply supported boundary conditions

conditions are defined as follows.
$\mathbf{S_3}$ **boundary condition**: Axially fixed, transversely free

$$w_0 = 0\,,\ M_x = 0\,,\ u_0 = 0\,,\ N_{xy} = 0 \tag{4.39}$$

S_4 boundary condition: Axially free, transversely free

$$w_0 = 0\ ,\ M_x = 0\ ,\ N_x = 0\ ,\ N_{xy} = 0 \tag{4.40}$$

The reader is refered to [4] for three-dimensional views of the four different simply supported boundary conditions. Similar simply supported boundary conditions can be prescribed along the edges $y = 0$ and $y = b$.

5 Cylindrical Bending of Laminated Plates

5.1 Governing equations

We consider a laminated composite plate of width a in the x direction. The plate is assumed to be infinitely long in the y direction and uniformly supported on the edges $x = 0$ and $x = a$ as shown in Fig. 5.1. The loads are assumed to be independent of y, i.e., $q = q(x,t)$. In this case, the laminated plate will deform into a cylindrical shape.

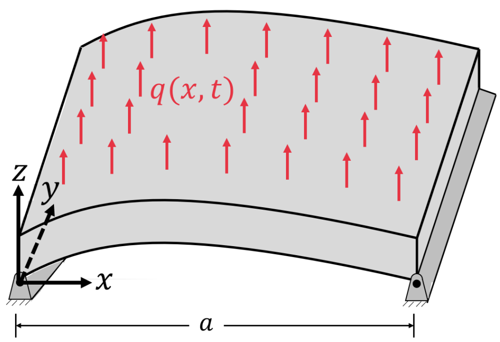


Figure 5.1: Cylindrical bending of a laminated plate

5.1.1 Displacements and strains

Since the loads and boundary conditions are independent of y, the resulting displacements, strains and stresses are independent of the y coordinate, i.e., $\frac{\partial(\cdot)}{\partial y} = 0$. In the case of cylindrical bending, the mid-surface displacements are functions of the x coordinate and time t. That is,

$$u_0 = u_0(x,t) \quad , \quad v_0 = v_0(x,t) \quad , \quad w_0 = w_0(x,t) \tag{5.1}$$

In the case of static loads,

$$u_0 = u_0(x) \quad , \quad v_0 = v_0(x) \quad , \quad w_0 = w_0(x) \tag{5.2}$$

The mid-surface strains and curvatures follow from (3.12), (3.13) and (5.2),

$$\varepsilon_x^0 = \frac{du_0}{dx}\,, \quad \varepsilon_y^0 = \frac{\cancelto{0}{dv_0}}{dy} = 0\,, \quad \gamma_{xy}^0 = \frac{\cancelto{0}{\partial u_0}}{\partial y} + \frac{\partial v_0}{\partial x} = \frac{dv_0}{dx} \tag{5.3a}$$

$$\kappa_x = -\frac{d^2 w_0}{dx^2}\,, \quad \kappa_y = -\frac{\partial^2 w_0}{\partial y^2} = 0\,, \quad \kappa_{xy} = -2\frac{\partial^2 w_0}{\partial x \partial y} = 0 \tag{5.3b}$$

5.1.2 Force and moment resultants

The force ans moment resultants follow from (3.26) and (5.3)

$$N_x = A_{11}\varepsilon_x^0 + A_{16}\gamma_{xy}^0 + B_{11}\kappa_x = A_{11}\frac{du_0}{dx} + A_{16}\frac{dv_0}{dx} - B_{11}\frac{d^2 w_0}{dx^2} \tag{5.4a}$$

$$N_y = A_{12}\frac{du_0}{dx} + A_{26}\frac{dv_0}{dx} - B_{12}\frac{d^2 w_0}{dx^2} \tag{5.4b}$$

$$N_{xy} = A_{16}\frac{du_0}{dx} + A_{66}\frac{dv_0}{dx} - B_{16}\frac{d^2 w_0}{dx^2} \tag{5.4c}$$

$$M_x = B_{11}\frac{du_0}{dx} + B_{16}\frac{dv_0}{dx} - D_{11}\frac{d^2 w_0}{dx^2} \tag{5.4d}$$

$$M_y = B_{12}\frac{du_0}{dx} + B_{26}\frac{dv_0}{dx} - D_{12}\frac{d^2 w_0}{dx^2} \tag{5.4e}$$

$$M_{xy} = B_{16}\frac{du_0}{dx} + B_{66}\frac{dv_0}{dx} - D_{16}\frac{d^2 w_0}{dx^2} \tag{5.4f}$$

5.1.3 Equilibrium Equations

The equilibrium equation (4.28a) reduces to

$$\frac{\partial N_x}{\partial x} + \frac{\cancelto{0}{\partial N_{xy}}}{\partial y} = 0 \quad \Rightarrow \quad \frac{dN_x}{dx} = 0 \tag{5.5}$$

Substituting for N_x from (5.4a) into (5.5) gives

$$A_{11}\frac{d^2 u_0}{dx^2} + A_{16}\frac{d^2 v_0}{dx^2} - B_{11}\frac{d^3 w_0}{dx^3} = 0 \tag{5.6}$$

The equilibrium equation (4.28b) reduces to

$$\frac{\partial N_{xy}}{\partial x} + \cancelto{0}{\frac{\partial N_y}{\partial y}} = 0 \quad \Rightarrow \quad \frac{dN_{xy}}{dx} = 0 \tag{5.7}$$

Substituting for N_{xy} from (5.4c) into (5.7) gives

$$A_{16}\frac{d^2u_0}{dx^2} + A_{66}\frac{d^2v_0}{dx^2} - B_{16}\frac{d^3w_0}{dx^3} = 0 \tag{5.8}$$

The equilibrium equation (4.28c) reduces to

$$\frac{\partial^2 M_x}{\partial x^2} + 2\cancelto{0}{\frac{\partial^2 M_{xy}}{\partial x \partial y}} + \cancelto{0}{\frac{\partial^2 M_y}{\partial y^2}} + q\,(x, y) = 0 \quad \Rightarrow \quad \frac{d^2M_x}{dx^2} + q\,(x) = 0 \tag{5.9}$$

Substituting for M_x from (5.4d) into (5.9) gives

$$D_{11}\frac{d^4w_0}{dx^4} - B_{11}\frac{d^3u_0}{dx^3} - B_{16}\frac{d^3v_0}{dx^3} = q\,(x) \tag{5.10}$$

Equations (5.6), (5.8) and (5.10) are the equilibrium equations for cylindrical bending expressed in terms of the mid-surface displacements u_0, v_0 and w_0.

5.2 General solution for cylindrical bending

The equilibrium equations (5.6) and (5.8) can be expressed in matrix form as

$$\begin{bmatrix} A_{11} & A_{16} \\ A_{16} & A_{66} \end{bmatrix} \begin{Bmatrix} \frac{d^2u_0}{dx^2} \\ \frac{d^2v_0}{dx^2} \end{Bmatrix} = \begin{Bmatrix} B_{11} \\ B_{16} \end{Bmatrix} \frac{d^3w_0}{dx^3} \tag{5.11}$$

Solving the system in (5.11) one obtains

$$\begin{Bmatrix} \frac{d^2u_0}{dx^2} \\ \frac{d^2v_0}{dx^2} \end{Bmatrix} = \frac{1}{\tilde{A}} \begin{bmatrix} A_{66} & -A_{16} \\ -A_{16} & A_{11} \end{bmatrix} \begin{Bmatrix} B_{11} \\ B_{16} \end{Bmatrix} \frac{d^3w_0}{dx^3} \tag{5.12}$$

which gives

$$\frac{d^2u_0}{dx^2} = \frac{\tilde{B}}{\tilde{A}}\frac{d^3w_0}{dx^3} \tag{5.13a}$$

$$\frac{d^2v_0}{dx^2} = \frac{\tilde{C}}{\tilde{A}}\frac{d^3w_0}{dx^3} \tag{5.13b}$$

where the constants $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ in (5.13) are related to the laminate rigidities as follows,

$$\tilde{A} = A_{11}A_{66} - A_{16}^2 \tag{5.14a}$$

$$\tilde{B} = A_{66}B_{11} - A_{16}B_{16} \tag{5.14b}$$

$$\tilde{C} = A_{11}B_{16} - A_{16}B_{11} \tag{5.14c}$$

Differentiating the equations in (5.13) and substituting into (5.10) we obtain

$$D_{11}\frac{d^4w_0}{dx^4} - B_{11}\frac{\tilde{B}}{\tilde{A}}\frac{d^4w_0}{dx^4} - B_{16}\frac{\tilde{C}}{\tilde{A}}\frac{d^4w_0}{dx^4} = q\left(x\right) \tag{5.15}$$

which can be factored and written as

$$\tilde{D}\ \frac{d^4w_0}{dx^4} = q\left(x\right) \tag{5.16}$$

where $\tilde{D}$ is defined as

$$\tilde{D}= D_{11} - B_{11}\frac{\tilde{B}}{\tilde{A}} - B_{16}\frac{\tilde{C}}{\tilde{A}} \tag{5.17}$$

Equation (5.16) can be integrated to obtain $\frac{d^3w_0}{dx^3}$ and the result substituted into the equations in (5.13) to obtain the differential equations for u_0 and v_0

$$\frac{d^2u_0}{dx^2} = \frac{\tilde{B}}{\tilde{A}}\cdot\frac{1}{\tilde{D}}\int q\left(x\right)dx \tag{5.18a}$$

$$\frac{d^2v_0}{dx^2} = \frac{\tilde{C}}{\tilde{A}}\cdot\frac{1}{\tilde{D}}\int q\left(x\right)dx \tag{5.18b}$$

5.3 Solution for uniformly distributed load

If the laminate is subjected to a uniform distributed load $q\left(x\right) = q_0$, then the integral of the load in equations (5.18a) and (5.18b) can be explicitly evaluated as

$$\int q\left(x\right)dx = \int q_0 dx = q_0 x + q_0 c_1 a = q_0\left(x + c_1 a\right) \tag{5.19}$$

where c_1 is a constant.

When the integral from (5.19) is substituted into (5.18a) and integrated as follows

$$\begin{aligned}\frac{d^2 u_0}{dx^2} &= \frac{\tilde{B}}{\tilde{A}\tilde{D}}\int q\left(x\right)dx = \frac{\tilde{B}\, q_0}{\tilde{A}\tilde{D}}\left(x + c_1 a\right)\\ \frac{du_0}{dx} &= \frac{\tilde{B}\, q_0}{\tilde{A}\tilde{D}}\left(\frac{x^2}{2} + c_1 a x + c_2 a^2\right)\end{aligned}$$

we obtain the following general form for the mid-surface displacement u_0

$$u_0 = \frac{\tilde{B}\, q_0}{\tilde{A}\tilde{D}}\left(\frac{x^3}{6} + \frac{c_1 a x^2}{2} + c_2 a^2 x + c_3 a^3\right) \tag{5.20}$$

where c_2 and c_3 are integration constants. Similarly, the integral (5.19) is substituted into (5.18b) and integrated

$$\begin{aligned}\frac{d^2 v_0}{dx^2} &= \frac{\tilde{C}}{\tilde{A}\tilde{D}}\int q\left(x\right)dx = \frac{\tilde{C}\, q_0}{\tilde{A}\tilde{D}}\left(x + c_1 a\right)\\ \frac{dv_0}{dx} &= \frac{\tilde{C}\, q_0}{\tilde{A}\tilde{D}}\left(\frac{x^2}{2} + c_1 a x + c_4 a^2\right)\end{aligned}$$

to obtain the following general form for the mid-surface displacement v_0

$$v_0 = \frac{\tilde{C}\, q_0}{\tilde{A}\tilde{D}}\left(\frac{x^3}{6} + \frac{c_1 a x^2}{2} + c_4 a^2 x + c_5 a^3\right) \tag{5.21}$$

where c_4 and c_5 are integration constants. Next, (5.16) in integrated four times

$$\begin{aligned}\tilde{D}\,\frac{d^3 w_0}{dx^3} &= \int q\left(x\right)dx = q_0\left(x + c_1 a\right)\\ \frac{d^2 w_0}{dx^2} &= \frac{q_0}{\tilde{D}}\left(\frac{x^2}{2} + c_1 a x + c_6 a^2\right)\\ \frac{dw_0}{dx} &= \frac{q_0}{\tilde{D}}\left(\frac{x^3}{6} + \frac{c_1 a x^2}{2} + c_6 a^2 x + c_7 a^3\right)\end{aligned}$$

to obtain the following general solution for the transverse deflection w_0

$$w_0 = \frac{q_0}{\tilde{D}}\left(\frac{x^4}{24} + \frac{c_1 a x^3}{6} + \frac{c_6 a^2 x^2}{2} + c_7 a^3 x + c_8 a^4\right) \tag{5.22}$$

where c_6, c_7 and c_8 are intergration constants. The eight integration constants $c_1, \ldots, c_8$ in (5.20), (5.21) and (5.22) are obtained by satisfying the relevant boundary conditions at $x = 0$ and $x = a$.

The force resultant N_x is obtained by substituting for u_0, v_0 and w_0 from (5.20), (5.21) and (5.22) into (5.4a),

$$\begin{aligned}
N_x &= A_{11}\frac{du_0}{dx} + A_{16}\frac{dv_0}{dx} - B_{11}\frac{d^2 w_0}{dx^2} \\
&= \frac{A_{11}\,\tilde{B}\,q_0}{\tilde{A}\tilde{D}}\left(\frac{x^2}{2} + c_1 a x + c_2 a^2\right) + \frac{A_{16}\,\tilde{C}\,q_0}{\tilde{A}\tilde{D}}\left(\frac{x^2}{2} + c_1 a x + c_4 a^2\right) - \frac{B_{11} q_0}{\tilde{D}}\left(\frac{x^2}{2} + c_1 a x + c_6 a^2\right) \\
&= \frac{q_0}{\tilde{A}\tilde{D}}\left(A_{11}\,\tilde{B} + A_{16}\,\tilde{C} - B_{11}\,\tilde{A}\right)\left(\frac{x^2}{2} + c_1 a x\right) + \frac{q_0 a^2}{\tilde{A}\tilde{D}}\left(A_{11}\,\tilde{B}\,c_2 + A_{16}\,\tilde{C}\,c_4 - B_{11}\,\tilde{A}\,c_6\right) \\
&= \cancelto{0}{\frac{q_0}{\tilde{A}\tilde{D}}\left(A_{11}\,\tilde{B} + A_{16}\,\tilde{C} - B_{11}\,\tilde{A}\right)}\left(\frac{x^2}{2} + c_1 a x\right) + \frac{q_0 a^2}{\tilde{A}\tilde{D}}\left(A_{11}\,\tilde{B}\,c_2 + A_{16}\,\tilde{C}\,c_4 - B_{11}\,\tilde{A}\,c_6\right)
\end{aligned} \tag{5.23a}$$

where it can be shown that the first term is zero based on the definitions for $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ in (5.14). Thus, the force resultant N_x reduces to

$$N_x = \frac{q_0 a^2}{\tilde{A}\tilde{D}}\left(A_{11}\,\tilde{B}\,c_2 + A_{16}\,\tilde{C}\,c_4 - B_{11}\,\tilde{A}\,c_6\right) \tag{5.24}$$

The force resultant N_{xy} is obtained by substituting for u_0, v_0 and w_0 from (5.20), (5.21) and (5.22) into (5.4c),

$$\begin{aligned}
N_{xy} &= A_{16}\frac{du_0}{dx} + A_{66}\frac{dv_0}{dx} - B_{16}\frac{d^2 w_0}{dx^2} \\
&= \cancelto{0}{\frac{q_0}{\tilde{A}\tilde{D}}\left(A_{16}\,\tilde{B} + A_{66}\,\tilde{C} - B_{16}\,\tilde{A}\right)}\left(\frac{x^2}{2} + c_1 a x\right) + \frac{q_0 a^2}{\tilde{A}\tilde{D}}\left(A_{16}\,\tilde{B}\,c_2 + A_{66}\,\tilde{C}\,c_4 - B_{16}\,\tilde{A}\,c_6\right)
\end{aligned}$$

which simplifies to

$$N_{xy} = \frac{q_0 a^2}{\tilde{A}\tilde{D}}\left(A_{16}\,\tilde{B}\,c_2 + A_{66}\,\tilde{C}\,c_4 - B_{16}\,\tilde{A}\,c_6\right) \tag{5.26}$$

The moment resultant M_x is obtained by substituting for u_0, v_0 and w_0 from (5.20), (5.21) and (5.22) into (5.4d),

$$
\begin{aligned}
M_x &= B_{11}\frac{du_0}{dx} + B_{16}\frac{dv_0}{dx} - D_{11}\frac{d^2w_0}{dx^2} \\
&= \frac{q_0}{\tilde{A}\tilde{D}}\left(B_{11}\,\tilde{B} + B_{16}\,\tilde{C} - D_{11}\,\tilde{A}\right)\left(\frac{x^2}{2} + c_1ax\right) + \frac{q_0a^2}{\tilde{A}\tilde{D}}\left(B_{11}\,\tilde{B}\,c_2 + B_{16}\,\tilde{C}\,c_4 - D_{11}\,\tilde{A}\,c_6\right) \\
&= -\frac{q_0}{\tilde{D}}\cancelto{\tilde{D}}{\left(D_{11} - B_{11}\frac{\tilde{B}}{\tilde{A}} - B_{16}\frac{\tilde{C}}{\tilde{A}}\right)}\left(\frac{x^2}{2} + c_1ax\right) + \frac{q_0a^2}{\tilde{A}\tilde{D}}\left(B_{11}\,\tilde{B}\,c_2 + B_{16}\,\tilde{C}\,c_4 - D_{11}\,\tilde{A}\,c_6\right)
\end{aligned}
$$

which simplifies to

$$
M_x = -q_0\left(\frac{x^2}{2} + c_1ax\right) + \frac{q_0a^2}{\tilde{A}\tilde{D}}\left(B_{11}\,\tilde{B}\,c_2 + B_{16}\,\tilde{C}\,c_4 - D_{11}\,\tilde{A}\,c_6\right) \tag{5.28}
$$

5.4 Simply supported laminated plate under uniform distributed load

Let's consider the cylindrical bending of a laminate that is simply supported at the edges and subjected to a uniform distributed load of magnitude q_0 in the positive z direction. The edge $x = 0$ is subjected to S_1 boundary conditions ($w_0 = 0,\ M_x = 0,\ u_0 = 0,\ v_0 = 0$) and the edge $x = a$ is subjected to S_4 boundary conditions ($w_0 = 0,\ M_x = 0,\ N_x = 0,\ N_{xy} = 0$).

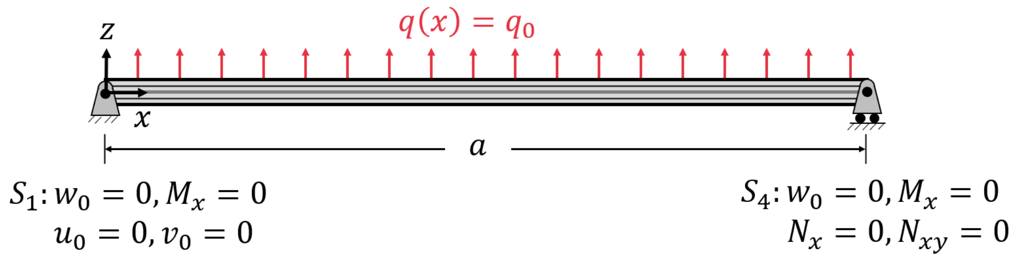


Figure 5.2: Cylindrical bending of a simply supported plate subjected to a uniform distributed load

5.4.1 Determination of constants

The eight integration constants $c_1, \ldots, c_8$ in (5.20), (5.21), (5.22), (5.24), (5.26) and (5.28) are obtained by satisfying the relevant boundary conditions at $x = 0$ and $x = a$.

Boundary conditions at $x = 0$

$$u_0(0) = 0 \quad \Rightarrow \quad c_3 = 0 \tag{5.29a}$$

$$v_0(0) = 0 \quad \Rightarrow \quad c_5 = 0 \tag{5.29b}$$

$$w_0(0) = 0 \quad \Rightarrow \quad c_8 = 0 \tag{5.29c}$$

$$M_x(0) = 0 \quad \Rightarrow \quad \frac{q_0 a^2}{\tilde{A}\tilde{D}} \left(B_{11}\, \tilde{B}\, c_2 + B_{16}\, \tilde{C}\, c_4 - D_{11}\, \tilde{A}\, c_6 \right) = 0$$

$$\Rightarrow \quad B_{11}\, \tilde{B}\, c_2 + B_{16}\, \tilde{C}\, c_4 - D_{11}\, \tilde{A}\, c_6 = 0 \tag{5.29d}$$

Boundary conditions at $x = a$

$$w_0(a) = 0 \quad \Rightarrow \quad \frac{q_0}{\tilde{D}} \left(\frac{a^4}{24} + \frac{c_1 a^4}{6} + \frac{c_6 a^4}{2} + c_7 a^4 + \cancelto{0}{c_8}\, a^4 \right) = 0$$

$$\Rightarrow \quad 1 + 4c_1 + 12c_6 + 24c_7 = 0 \tag{5.30a}$$

$$M_x(a) = 0 \quad \Rightarrow \quad -q_0 \left(\frac{a^2}{2} + c_1 a^2 \right) + \frac{q_0 a^2}{\tilde{A}\tilde{D}} \underbrace{\left(B_{11}\, \tilde{B}\, c_2 + B_{16}\, \tilde{C}\, c_4 - D_{11}\, \tilde{A}\, c_6 \right)}_{= 0 \text{ by equation (5.29d)}} = 0$$

$$\Rightarrow \quad \frac{a^2}{2} + c_1 a^2 = 0 \quad \Rightarrow \quad c_1 = -\frac{1}{2} \tag{5.30b}$$

$$N_x(a) = 0 \quad \Rightarrow \quad A_{11}\, \tilde{B}\, c_2 + A_{16}\, \tilde{C}\, c_4 - B_{11}\, \tilde{A}\, c_6 = 0 \tag{5.30c}$$

$$N_{xy}(a) = 0 \quad \Rightarrow \quad A_{16}\, \tilde{B}\, c_2 + A_{66}\, \tilde{C}\, c_4 - B_{16}\, \tilde{A}\, c_6 = 0 \tag{5.30d}$$

Equations (5.29d), (5.30a), (5.30c), and (5.30d) need to be solved to obtain $c_2,\ c_4,\ c_6$ and c_7. The solution $c_2 = c_4 = c_6 = 0$ satisfies equations (5.29d), (5.30c), and (5.30d). By equation (5.30a):

$$1 + \cancelto{-\frac{1}{2}}{4c_1} + \cancelto{0}{12c_6} + 24c_7 = 0 \quad \Rightarrow \quad c_7 = \frac{1}{24}$$

In summary, the constants c_i are

$$c_1 = -\frac{1}{2}\ ;\ c_2 = c_3 = c_4 = c_5 = c_6 = 0\ ;\ c_7 = \frac{1}{24}\ ;\ c_8 = 0$$

5.4.2 Mid-surface displacements

We obtain the final solutions for the mid-surface displacements by substituting for the constants c_i into (5.20), (5.21) and (5.22). The mid-surface displacement u_0 simplifies to

$$u_0(x) = \frac{\tilde{B}\ q_0}{\tilde{A}\tilde{D}}\left(\frac{x^3}{6} + \cancelto{-\frac{1}{2}}{c_1} a\frac{x^2}{2} + \cancelto{0}{c_2} a^2 x + \cancelto{0}{c_3} a^3\right)$$

$$= \frac{\tilde{B}\ q_0 x^2}{12\ \tilde{A}\tilde{D}}\,(2x - 3a) \tag{5.31a}$$

The mid-surface displacement v_0 simplifies to

$$v_0(x) = \frac{\tilde{C}\ q_0}{\tilde{A}\tilde{D}}\left(\frac{x^3}{6} + \cancelto{-\frac{1}{2}}{c_1} a\frac{x^2}{2} + \cancelto{0}{c_4} a^2 x + \cancelto{0}{c_5} a^3\right)$$

$$= \frac{\tilde{C}\ q_0 x^2}{12\ \tilde{A}\tilde{D}}\,(2x - 3a) \tag{5.32a}$$

and the mid-surface displacement w_0 becomes

$$w_0(x) = \frac{q_0}{\tilde{D}}\left(\frac{x^4}{24} + \frac{\cancelto{-\frac{1}{2}}{c_1} a x^3}{6} + \frac{\cancelto{0}{c_6} a^2 x^2}{2} + \cancelto{\frac{1}{24}}{c_7} a^3 x + \cancelto{0}{c_8} a^4\right)$$

$$= \frac{q_0 x}{24\ \tilde{D}}\left(x^3 - 2ax^2 + a^3\right) \tag{5.33a}$$

5.4.3 Mid-surface strains and curvatures

Once we have $u_0\,(x)\,,\ v_0\,(x)\,,\ w_0\,(x)$ the mid-surface strains and curvatures can be calculated at any location. The mid-surface strains strains are

$$\varepsilon_x^0 = \frac{du_0}{dx} = \frac{d}{dx}\left[\frac{\tilde{B}\ q_0}{12\ \tilde{A}\tilde{D}}\left(2x^3 - 3ax^2\right)\right] = \frac{\tilde{B}\ q_0}{2\ \tilde{A}\tilde{D}}x\,(x - a) \tag{5.34a}$$

$$\varepsilon_y^0 = 0 \tag{5.34b}$$

$$\gamma_{xy}^0 = \frac{dv_0}{dx} = \frac{\tilde{C}\ q_0}{2\ \tilde{A}\tilde{D}}x\,(x - a) \tag{5.34c}$$

Note that the in-plane normal strain ε_x^0 and in-plane γ_{xy}^0 are not zero if the laminate constants $\tilde{B}$ and $\tilde{C}$ are non-zero. The curvatures reduce to

$$\kappa_x = -\frac{d^2 w_0}{dx^2} = -\frac{q_0}{2\,\tilde{D}} x\,(x-a) \tag{5.35a}$$

$$\kappa_y = 0 \tag{5.35b}$$

$$\kappa_{xy} = 0 \tag{5.35c}$$

5.4.4 Force and moment resultants

Once the constants $c_1, \ldots, c_8$ have been determined from the boundary conditions, the force and moment resultants are obtained from (5.24), (5.26) and (5.28),

$$N_x(x) = 0, \quad N_{xy}(x) = 0, \quad M_x(x) = \frac{q_0}{2} x(a-x) \tag{5.36}$$

It makes sense that the force resultants N_x and N_{xy} are zero since the simple support at $x = a$ is subjected to S_4 boundary conditions wherein $N_x(a) = 0$ and $N_{xy}(a) = 0$. Depending on the stacking sequence, the force resultant N_x may be non-zero if the support at $x = a$ is restrained against axial displacement.

5.4.5 Maximum deflection

The maximum deflection $w_0^{\max}$ occurs at the mid-span $x = a/2$

$$w_0^{\max} = w_0\left(\frac{a}{2}\right) = \frac{q_0 \frac{a}{2}}{24\,\tilde{D}}\left(\frac{a^3}{8} - \frac{a^3}{2} + a^3\right) = \frac{5 q_0 a^4}{384\,\tilde{D}} \tag{5.37}$$

Recall that, $\tilde{D}= D_{11} - B_{11}\frac{\tilde{B}}{\tilde{A}} - B_{16}\frac{\tilde{C}}{\tilde{A}}$. Therefore, the in-plane extensional rigidities A_{ij} and the bending-extension rigidities B_{ij} also influence the transverse deflection!

If the deflection is calculated by neglecting the elastic coupling rigidities (i.e., by setting $\tilde{B}= 0$, $\tilde{C}= 0$), then the maximum deflection $\hat{w}_0^{\max}$ depends only on the bending rigidity D_{11} since

$$\hat{w}_0^{\max} = \frac{5 q_0 a^4}{384 D_{11}} \tag{5.38}$$

Therefore, the ratio of the maximum deflections can be expressed as

$$\frac{w_0^{\max}}{\hat{w}_0^{\max}} = \frac{D_{11}}{\tilde{D}} = \frac{\tilde{D} + B_{11}\frac{\tilde{B}}{\tilde{A}} + B_{16}\frac{\tilde{C}}{\tilde{A}}}{\tilde{D}} = 1 + \frac{B_{11}\,\tilde{B}}{\tilde{A}\tilde{D}} + \frac{B_{16}\,\tilde{C}}{\tilde{A}\tilde{D}} = 1 + \tilde{E} \tag{5.39}$$

where $\tilde{E} = \frac{B_{11}\tilde{B}}{\tilde{A}\tilde{D}} + \frac{B_{16}\tilde{C}}{\tilde{A}\tilde{D}}$. Thus,

$$w_0^{\max} = \left(1 + \tilde{E}\right)\hat{w}_0^{\max} \tag{5.40}$$

It can be shown that $\tilde{E}$ is always positive. Therefore, the elastic coupling rigidities tends to increase the maximum deflection of the laminate.

5.4.6 In-plane displacements at the support $x = a$

The axial displacement of the laminate at the right support (i.e. at $x = a$) is

$$u_0(a) = -\frac{\tilde{B}\,q_0 a^3}{12\,\tilde{A}\tilde{D}} \tag{5.41}$$

The axial displacement can be positive or negative depending on the direction of q_0 and the sign of the coupling coefficients. Note that in the case of a laminate, the axial displacement, is a linear effect. The axial displacement will be in the opposite direction if the direction of the applied load is reversed. This is qualitatively different from curvature shortening which is a non-linear effect. In the case of curvature shortening, the right support will move inward by an amount λ where

$$\lambda = \frac{8\delta^2}{3a} \tag{5.42}$$

where δ is the maximum deflection. Thus, the curvature shortening λ is always positive and is proportional to the square of the maximum deflection, i.e., this is not a linear effect, unlike what we see in laminated composites.

It is noted that in the case of a laminated composite plate, the right support can displace in the y direction if the coefficient $\tilde{C}$ is non-zero since

$$v_0(a) = -\frac{\tilde{C}\,q_0 a^3}{12\,\tilde{A}\tilde{D}} \tag{5.43}$$

EXAMPLE 5.1: Cylindrical bending of a simply supported laminated plate subjected to a uniform distributed load

Let's consider the cylindrical bending of a unidirectional IM7/8552 carbon fiber-reinforced $[0_2/90_2]$ laminated plate of length $a = 0.5$ m that is simply supported at the edges as shown in Fig. 5.3. The laminated plate is subjected to a distributed load of magnitude 1 N/m^2, i.e., $q_0 = -1$ N/m^2. The edge $x = 0$ is subjected to S_1 boundary conditions ($w_0 = 0,\ M_x = 0,\ u_0 = 0,\ v_0 = 0$) and the edge $x = a$ is subjected to S_4 boundary conditions ($w_0 = 0,\ M_x = 0,\ N_x = 0,\ N_{xy} = 0$). The laminate thickness $H = 0.8$ mm.

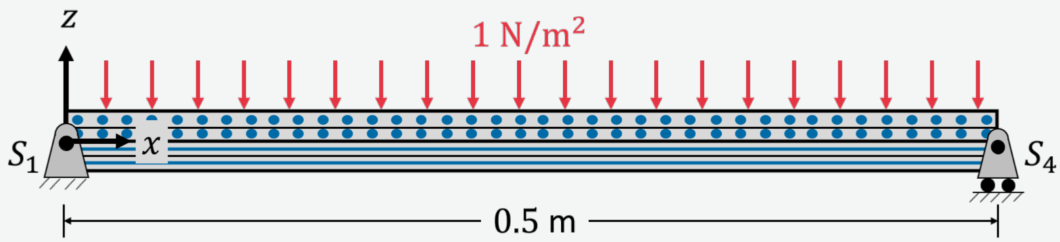


Figure 5.3: Cylindrical bending of a $[0_2/90_2]$ unidirectional fiber reinforced laminate subjected to S_1-S_4 simply supported boundary conditions

The relevant elastic rigidities of the $[0_2/90_2]$ laminate are

$$A_{11} = 71.20 \times 10^6 \text{ N/m}, \quad A_{66} = 3.84 \times 10^6 \text{ N/m}, \quad B_{11} = -1.271 \times 10^4 \text{ N}, \quad D_{11} = 3.80 \text{ N} \cdot \text{m} \tag{5.44}$$

Note that the elastic rigidities A_{16} and B_{16} are identically zero for the cross-ply laminate. The constants $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$ are evaluated using (5.14) and (5.17),

$$\tilde{A} = 2.73 \times 10^{14} \text{ N}^2/\text{m}^2, \quad \tilde{B} = -4.88 \times 10^{10} \text{ N}^2/\text{m}, \quad \tilde{C} = 0 \text{ N}^2/\text{m}, \quad \tilde{D} = 1.528 \text{ N} \cdot \text{m} \tag{5.45}$$

The deflection $w_0(x)$ of the laminated plate is evaluated using (5.33a) and is shown in Fig. 5.4.

The maximum deflection of the plate

$$w_0^{\max} = \frac{5 q_0 a^4}{384\, \tilde{D}} = -0.532 \text{ mm} \tag{5.46}$$

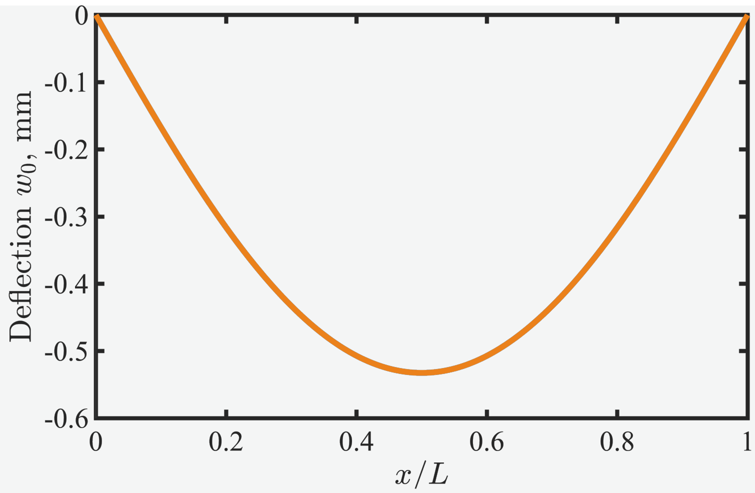


Figure 5.4: Deflection w_0 of a $[0_2/90_2]$ laminate subjected to S_1-S_4 simply supported boundary conditions

Since the laminate is unsymmetric, the bending-extension coupling rigiditiy B_{11} induces an in-plane displacement u_0 that is shown in Fig. 5.5.

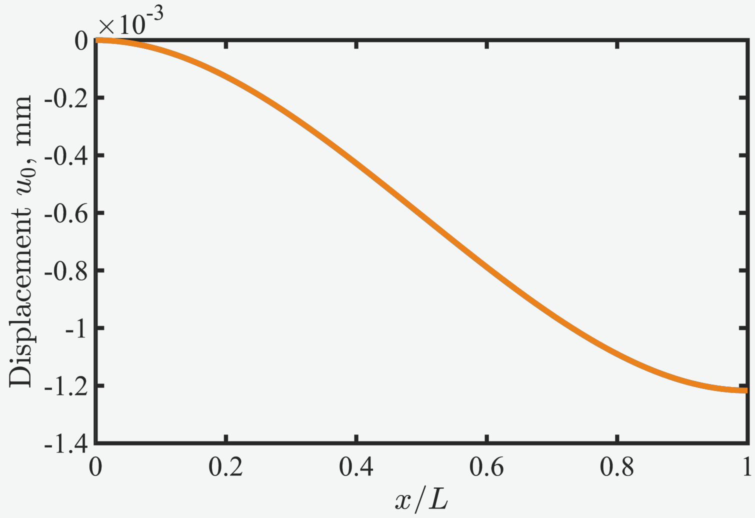


Figure 5.5: Mid-surface axial displacement u_0 of a $[0_2/90_2]$ laminate subjected to S_1-S_4 simply supported boundary conditions

The axial displacement $u_0(a)$ at the right support is

$$u_0(a) = -\frac{\widetilde{B}\ q_0 a^3}{12\ \widetilde{A}\widetilde{D}} = -1.217 \times 10^{-3}\ \text{mm} \tag{5.47}$$

Although the axial displacement u_0 is small relative to w_0, it cannot be ignored in our analysis. The through-thickness variation of the normal strain ε_x and the normal stress σ_x in the global coordinate system are shown in Fig. 5.6. As expected, the normal stress σ_x is tensile on the bottom surface and compressive on the top surface. However, portions of the second ply (0° lamina) are subjected to a compressive normal stress σ_x although it lies below the mid-surface of the laminate.

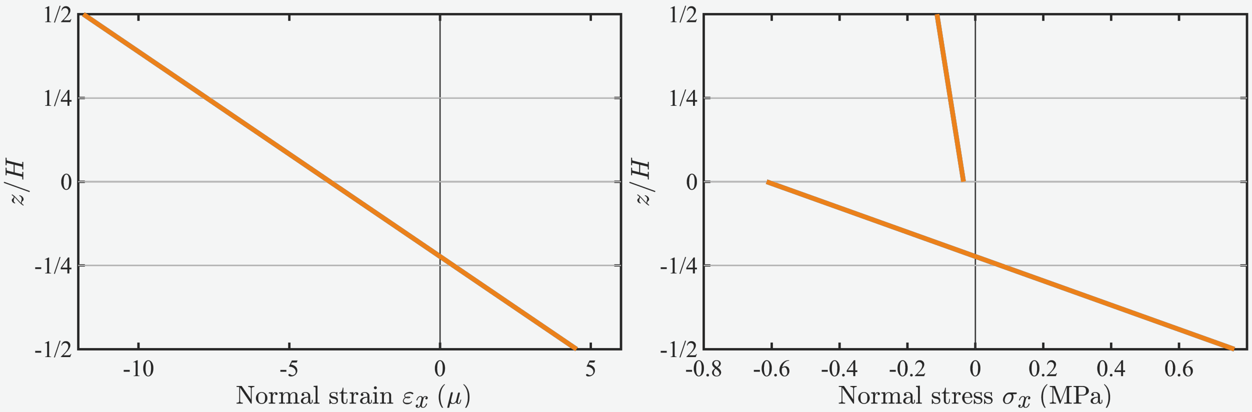


Figure 5.6: Normal strain ε_x and normal stress σ_x at $x = a/2$ for a $[0_2/90_2]$ laminate subjected to S_1-S_4 simply supported boundary conditions

The through-thickness variation of the longitudinal normal stress σ_1 and the transverse normal stress σ_2 in the principal material coordinate system are shown in Fig. 5.7.

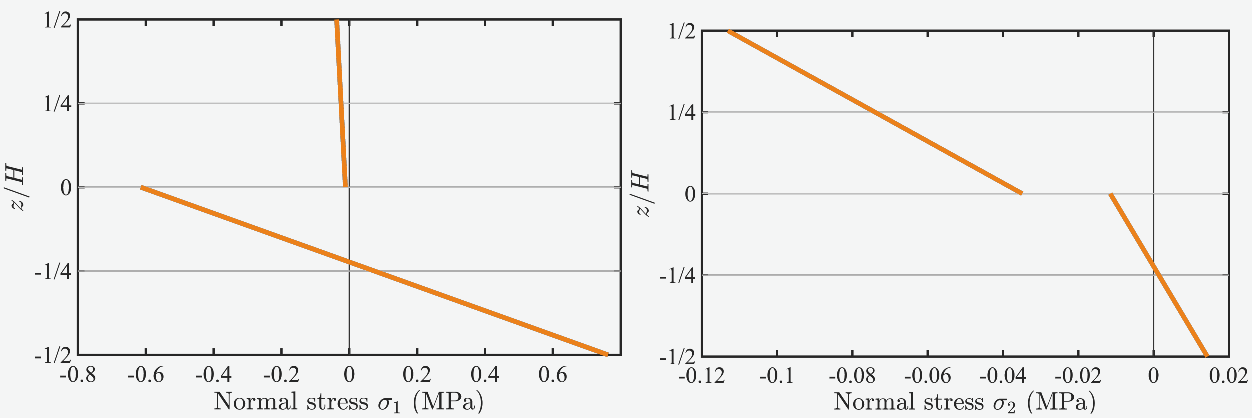


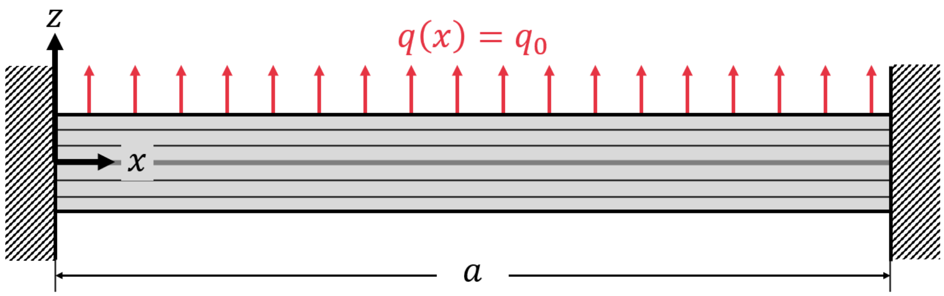
Figure 5.7: Normal stresses σ_1 and σ_2 at $x = a/2$ for a $[0_2/90_2]$ laminate subjected to S_1-S_4 simply supported boundary conditions

The largest normal stress s_1 in the fiber direction occurs on the bottom surface. The 90° plies are subjected to a compressive transverse normal stress s_2 with the largest value on the top surface. The minimum safety factor $S_{fa}^{\min} = 1785$ and it occurs on the top surface of the laminate. Note that although the laminate will not fail due to the large safety factor, the transverse deflection $w_0^{\max}$ is fairly large relative to the thickness of the laminate.

Exercises

Use the representative properties in Sec. 1.10.1 for unidirectional carbon/epoxy composites and the representative properties in Sec. 1.10.2 for fabric-reinforced carbon/epoxy composites. Assume that the ply thickness h of the unidirectional and fabric-reinforced laminae are 0.2 mm and 0.4 mm, respectively. Use the Tsai-Wu theory for failure analysis.

5.1 Consider the cylindrical bending of a laminated composite plate of length a that is clamped on both edges (i.e., $w_0 = \partial w_0/\partial x = u_0 = v_0 = 0$ at $x = 0$ and $x = a$). The laminate is subjected to a uniform distributed load $q(x) = q_0$.

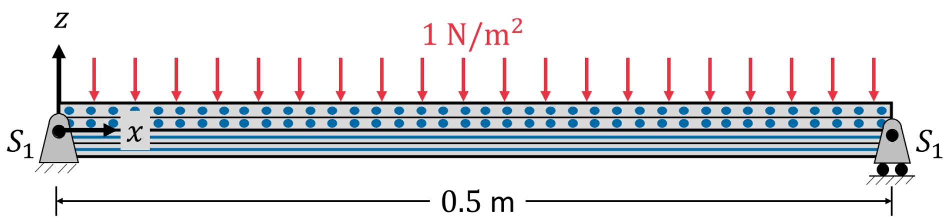


(a) Evaluate the constants $c_1, c_2, \ldots, c_8$ in the general solution for a uniformly distributed load using the boundary conditions.

(b) Provide analytical expressions for the mid-surface displacements $u_0(x)$, $v_0(x)$ and $w_0(x)$, the mid-surface strains $\varepsilon_x^0(x)$ and $\gamma_{xy}^0(x)$, the curvature $\kappa_x(x)$ and the bending moment resultant $M_x(x)$.

(c) Analyze the cylindrical bending of a unidirectional carbon fiber-reinforced [0/90/45] laminate that is clamped on both edges. The length of the laminate $a = 0.5$ m and it is subjected to a downward uniform distributed load of magnitude 1 N/m^2. Plot the mid-surface displacements $u_0(x)$ and $v_0(x)$, the deflection $w_0(x)$ and the bending moment $M_x(x)$. Do the plots make sense? Plot the through-thickness variation of the strain ε_x and the stresses σ_x and τ_{xy} at $x = 0$ and $x = a/2$. Discuss whether the direction and magnitude of the normal stress σ_x makes sense at the clamped edge and at the mid-span.

5.2 Consider the cylindrical bending of a *cross-ply* laminate of length a. It is subjected to a uniform distributed load $q(x) = q_0$ and the edges at $x = 0$ and $x = a$ are both subjected to S_1 boundary conditions ($w_0 = 0, M_x = 0, u_0 = v_0 = 0$). Evaluate the constants $c_1, c_2, \ldots, c_8$ in the general solution for a uniformly distributed load using the boundary conditions. Analyze the cylindrical bending of a unidirectional carbon fiber-reinforced $[0_2/90_2]$ laminate. The length of the laminate $a = 0.5$ m and it is subjected to a downward uniform distributed load of magnitude 1 N/m^2.



(a) Plot the deflection $w_0(x)$ and compare it with the deflection obtained in Example 5.1 of the lecture notes for S$_1$-S$_4$ boundary conditions. Does the axial boundary conditions at $x = a$ influence the transverse deflection of the laminated plate? If it does, explain why.

(b) Plot the mid-surface displacement $u_0(x)$. Does it satisfy the S$_1$ boundary conditions at $x = 0$ and $x = a$? Discuss the variation of $u_0(x)$.

(c) Determine the in-place force resultant $N_x(x)$. Is it non-zero? If it is, can you explain why?

(d) Plot the through-thickness variation of the strain ε_x and the stress σ_x at the mid-span $x = a/2$. Discuss whether the direction and magnitude of the normal stress σ_x makes sense.

Navier Solution for Bending of Rectangular Plates 6

In this chapter, we derive analytical solutions for the bending of simply supported laminated rectangular plates using the solution process originally introduced by Navier for isotropic rectangular plates.

6.1 Series representation of applied loads

Consider a rectangular laminated plate that is simply supported on all four edges and subjected to a transverse distributed load $q(x, y)$ as shown in Fig. 6.1.

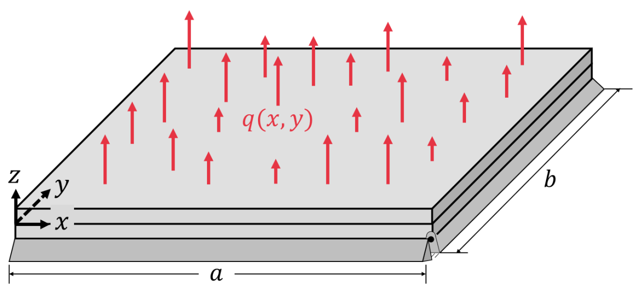


Figure 6.1: Simply supported laminated plate subjected to a distributed load

The distributed load $q(x, y)$ can be represented by the double Fourier series

$$q(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} q_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{6.1}$$

where the load coefficients q_{mn} are obtained by multiplying both sides of equation (6.1) by sine functions and double integrating as follows,

$$\begin{aligned}\int_0^b \int_0^a q(x, y) \sin \frac{k\pi x}{a} \sin \frac{l\pi y}{b} dxdy &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \int_0^b \int_0^a q_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \sin \frac{k\pi x}{a} \sin \frac{l\pi y}{b} dxdy \\ &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} q_{mn} \left(\int_0^a \sin \frac{m\pi x}{a} \sin \frac{k\pi x}{a} dx \right) \left(\int_0^b \sin \frac{n\pi y}{b} \sin \frac{l\pi y}{b} dy \right)\end{aligned} \tag{6.2}$$

where k and l are arbitrary integers. Since

$$\int_0^a \sin\frac{m\pi x}{a}\sin\frac{k\pi x}{a}dx = \begin{cases} 0 & \text{for } m \neq k \\ \frac{a}{2} & \text{for } m = k \end{cases} \tag{6.3}$$

it follows from equations (6.2) and (6.3)

$$\int_0^b\int_0^a q(x,y)\sin\frac{k\pi x}{a}\sin\frac{l\pi y}{b}dxdy = q_{kl}\left(\frac{a}{2}\right)\left(\frac{b}{2}\right) \tag{6.4}$$

Since k and l are arbitrary in (6.4), they can be replaced by m and n, respectively, to obtain

$$q_{mn} = \frac{4}{ab}\int_0^b\int_0^a q(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}dxdy \tag{6.5}$$

6.1.1 Uniform distributed load

Consider a laminated plate that is subjected to a uniform distributed load of magnitude q_0 acting vertically downward as shown in Fig. 6.2.

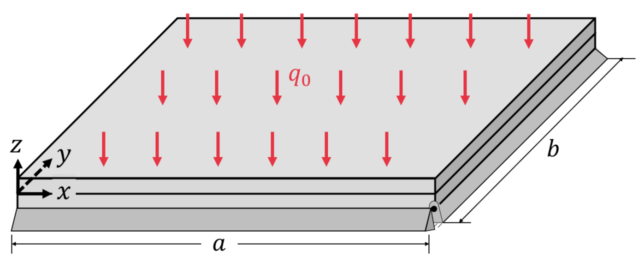


Figure 6.2: Simply supported laminated plate subjected to a uniform distributed load

In the case of a uniform distributed load

$$q(x,y) = -q_0 \tag{6.6}$$

The load coefficients q_{mn} for a uniform distributed are evaluated using (6.5) and (6.6)

$$q_{mn} = \frac{-4q_0}{ab}\left(-\frac{a}{m\pi}\cos\frac{m\pi x}{a}\right)\Bigg|_0^a \cdot \left(-\frac{b}{n\pi}\cos\frac{n\pi y}{b}\right)\Bigg|_0^b \tag{6.7a}$$

$$= -\frac{4q_0}{\cancel{ab}}\left(\frac{\cancel{ab}}{mn\pi^2}\right)[(-1)^m - 1]\cdot[(-1)^n - 1] \tag{6.7b}$$

which reduces to

$$q_{mn} = -\frac{4q_0}{mn\pi^2}\left[(-1)^m - 1\right] \cdot \left[(-1)^n - 1\right] \qquad (m, n = 1, 2, 3, \ldots) \tag{6.8}$$

Alternatively, since $[(-1)^m - 1]$ equals -2 when m is odd and 0 when m is even, the load coefficients q_{mn} can be written as

$$q_{mn} = \begin{cases} -\frac{16q_0}{mn\pi^2} & \text{for} \quad m, n = 1, 3, 5, \ldots \text{ (odd } m \text{ and } n\text{)} \\ 0 & \text{for} \quad m, n = 2, 4, 6, \ldots \text{ (even } m \text{ or } n\text{)} \end{cases} \tag{6.9}$$

6.1.2 Point load

Consider a laminated plate that is subjected to a point load of magnitude P acting downwards at (x_0, y_0) as shown in Fig. 6.3.

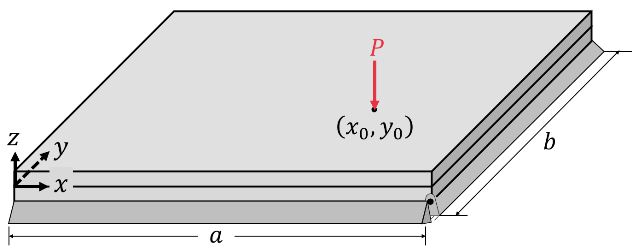


Figure 6.3: Simply supported laminated plate subjected to a point load

The point load can be represented by a Dirac delta function:

$$q(x, y) = -P\delta(x_0, y_0) \tag{6.10}$$

where δ is the Dirac delta function. The load coefficients q_{mn} for a point load are evaluated using (6.5) and (6.10)

$$q_{mn} = \frac{4}{ab}\int_0^b \int_0^a (-P)\,\delta(x_0, y_0) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} dx dy \tag{6.11}$$

Using the sifting property of the Dirac delta function, it follows from (6.11) that the load coefficients q_{mn} for a point load are

$$q_{mn} = -\frac{4P}{ab} \sin\frac{m\pi x_0}{a} \sin\frac{n\pi y_0}{b} \qquad (m, n = 1, 2, 3, \ldots) \tag{6.12}$$

6.2 Bending of specially orthotropic laminated plates

In this section, we derived an analytical solution for the bending of specially orthotropic rectangular laminated plates that are simply supported on all four edges. The plate is subjected to S_1 boundary conditions on all four edges as shown in Fig. 6.4. Specifically,

$$\begin{aligned} &w_0 = 0\,,\; M_x = 0\,,\; u_0 = 0\,,\; v_0 = 0 \quad \text{at} \quad x = 0,\; a \\ &w_0 = 0\,,\; M_y = 0\,,\; u_0 = 0\,,\; v_0 = 0 \quad \text{at} \quad y = 0,\; b \end{aligned} \tag{6.13}$$

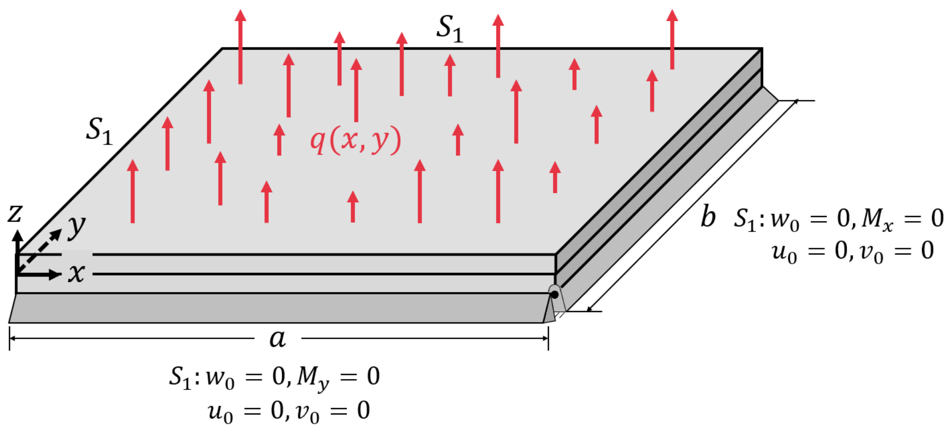


Figure 6.4: Simply supported laminated plate subjected S_1 boundary conditions on all four edges

A laminate is said to be specially orthotropic if

$$[B] = [0]\,, \quad A_{16} = A_{26} = 0, \quad D_{16} = D_{26} = 0 \tag{6.14}$$

<u>Examples of specially orthotropic laminates</u>

1. Symmetric cross-ply laminates, e.g., $[0/90/0]$, $[0/90]_s$ . . . etc.
2. There are some uncommon angle ply laminates that behave like specially orthotropic laminates, e.g., $[(\pm\theta)_s/(\mp\theta)_s]$ such as $[30/-30/-30/30/-30/30/30/-30]$. This is a special type of anti-symmetric balanced laminates.

6.2.1 Midsurface displacements

Specially orthotropic laminates do not exhibit bending-extension coupling since $[B] = [0]$. Therefore, when the laminate is subjected to bending, the mid-surface strains will be zero. Therefore, it is assumed that the mid-surface in-plane displacements u_0 and v_0 are zero and the deflection w_0 is a function of x and y, i.e.,

$$u_0\,(x, y) = v_0\,(x, y) = 0, \quad w_0 = w_0\,(x, y) \tag{6.15}$$

6.2.2 Midsurface strains and curvatures

Substituting for the assumed displacements u_0, v_0 and w_0 into (3.12) and (3.13), yields the following expressions for the mid-surface strains and curvatures

$$\varepsilon_x^0 = 0, \quad \varepsilon_y^0 = 0, \quad \gamma_{xy}^0 = 0 \tag{6.16}$$

$$\kappa_x = -\frac{\partial^2 w_0}{\partial x^2}, \quad \kappa_y = -\frac{\partial^2 w_0}{\partial y^2}, \quad \kappa_{xy} = -2\frac{\partial^2 w_0}{\partial x \partial y} \tag{6.17}$$

6.2.3 Force and moment resultants

Substituting for the mid-surface strains (6.16), curvatures (6.17) and laminate rigidities (6.14) into (3.21)) and (3.24) gives the force resultants

$$N_x = 0, \quad N_y = 0, \quad N_{xy} = 0 \tag{6.18}$$

and moment resultants

$$\begin{aligned} M_x &= -D_{11}\frac{\partial^2 w_0}{\partial x^2} - D_{12}\frac{\partial^2 w_0}{\partial y^2} \\ M_y &= -D_{12}\frac{\partial^2 w_0}{\partial x^2} - D_{22}\frac{\partial^2 w_0}{\partial y^2} \\ M_{xy} &= -2D_{66}\frac{\partial^2 w_0}{\partial x \partial y} \end{aligned} \tag{6.19}$$

6.2.4 Equilibrium equations

The equilibrium equations (4.28a) and (4.28b), namely

$$\begin{aligned} \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \\ \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0 \end{aligned} \tag{6.20}$$

are identically satisfied since $N_x = N_y = N_{xy} = 0$. The equilibrium equation (4.28c), namely

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + q\,(x, y) = 0 \tag{6.21}$$

needs to be satisfied at every point in the laminated plate. Substituting for the bending moments from (6.19) into (6.21)

$$-D_{11}\frac{\partial^4 w_0}{\partial x^4} - D_{12}\frac{\partial^4 w_0}{\partial x^2 \partial y^2} - 4D_{66}\frac{\partial^4 w_0}{\partial x^2 \partial y^2} - D_{12}\frac{\partial^4 w_0}{\partial x^2 \partial y^2} - D_{22}\frac{\partial^4 w_0}{\partial y^4} + q\left(x, y\right) = 0 \tag{6.22}$$

which can be factored as

$$D_{11}\frac{\partial^4 w_0}{\partial x^4} + 2\left(D_{12} + 2D_{66}\right)\frac{\partial^4 w_0}{\partial x^2 \partial y^2} + D_{22}\frac{\partial^4 w_0}{\partial y^4} = q\left(x, y\right) \tag{6.23}$$

6.2.5 Navier solution

Equation (6.23) is a fourth-order partial differential equation for the deflection $w_0(x, y)$. The solution procedure, as suggested by Navier, involves assuming a double Fourier sine series expansion for the deflection

$$w_0\left(x, y\right) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{6.24}$$

where W_{mn} are the deflection coefficients. Substitution of (6.24) and the Fourier series expansion (6.1) for $q\left(x, y\right)$ into (6.23) gives

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{W_{mn}\left[D_{11}\left(\frac{m\pi}{a}\right)^4 + 2\left(D_{12} + 2D_{66}\right)\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + D_{22}\left(\frac{n\pi}{b}\right)^4\right] - q_{mn}\right\} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} = 0 \tag{6.25}$$

Since this equation has to hold true for arbitrary x and y, the term in braces must equal zero

$$W_{mn}\left[D_{11}\left(\frac{m\pi}{a}\right)^4 + 2\left(D_{12} + 2D_{66}\right)\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + D_{22}\left(\frac{n\pi}{b}\right)^4\right] = q_{mn} \tag{6.26}$$

from which it follows that

$$W_{mn} = \frac{q_{mn}}{D_{11}\left(\frac{m\pi}{a}\right)^4 + 2\left(D_{12} + 2D_{66}\right)\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + D_{22}\left(\frac{n\pi}{b}\right)^4} \tag{6.27}$$

Thus, the deflection W_{mn} can be determined from the load coefficients q_{mn}, the bending rigidities D_{11}, D_{22}, D_{12} and D_{66} and the length a and width b of the rectangular plate. Subsequently, the deflection $w_0(x, y)$ can be calculated at any point in the plate using (6.24).

The curvatures are obtained by substituting for the deflection $w_0(x, y)$ from (6.24) into (6.17)

$$\begin{aligned}
\kappa_x &= -\frac{\partial^2 w_0}{\partial x^2} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{mn}\left(\frac{m\pi}{a}\right)^2 \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\\
\kappa_y &= -\frac{\partial^2 w_0}{\partial y^2} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{mn}\left(\frac{n\pi}{b}\right)^2 \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\\
\kappa_{xy} &= -2\frac{\partial^2 w_0}{\partial x\partial y} = -2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{mn}\left(\frac{mn\pi^2}{ab}\right) \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}
\end{aligned} \tag{6.28}$$

The through-thickness variations of strains and stresses can be obtained at any location (x, y) based on the mid-surface strains (6.16) and curvatures (6.28) at that location using (3.16).

The infinite series (6.24) is usually truncated to a finite number of terms when finding the displacements, strains and stresses, as

$$w_0(x, y) = \sum_{m=1}^{N_s}\sum_{n=1}^{N_s} W_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{6.29}$$

where N_s defines the number of terms retained in the series. The convergence of the strains and the stresses is typically slower than the deflection since they involve the derivatives of the deflection with respect to the spatial coordinates. Therefore, it is important to make sure that a sufficient number of terms have been used to obtain the strains and stresses accurately.

6.3 Bending of cross-ply laminated plates

In this section, we develop analytical solutions for cross-ply laminates that may exhibit bending-extension coupling. Consider simply supported laminates that are subjected to S_2 boundary conditions on all four edges as shown in Fig. 6.5. on all four edges as shown in Fig. 6.4. Specifically,

$$\begin{aligned}
w_0 = 0\,,\; M_x = 0\,,\; N_x = 0\,,\; v_0 = 0 \quad &\text{at} \quad x = 0,\; a\\
w_0 = 0\,,\; M_y = 0\,,\; u_0 = 0\,,\; N_y = 0 \quad &\text{at} \quad y = 0,\; b
\end{aligned} \tag{6.30}$$

In the case of a cross ply laminate, $\overline{Q}_{16} = \overline{Q}_{26} = 0$ for all lamina. Therefore,

$$A_{16} = A_{26} = 0, \quad B_{16} = B_{26} = 0, \quad D_{16} = D_{26} = 0 \tag{6.31}$$

The transverse loading $q(x, y)$ is expanded

$$q(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} q_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{6.32}$$

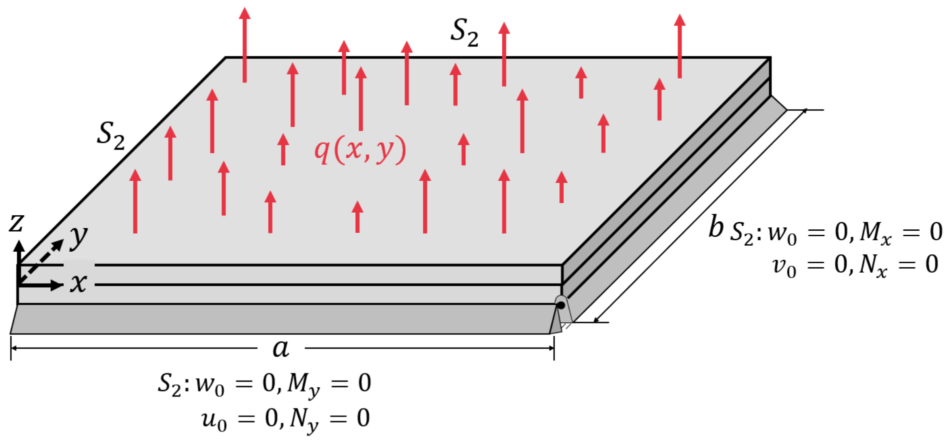


Figure 6.5: Simply supported laminated plate subjected S_2 boundary conditions on all four edges

where the load coefficients q_{mn} are evaluated using (6.5) and are listed in (6.8) and (6.12) for a uniform distributed load and a point load, respectively.

6.3.1 Midsurface displacements

In the case of unsymmetric cross-ply laminates, bending will induce in-plane mid-surface extensional strains and displacements u_0 and v_0. The solution procedure, suggested by Navier, involves assuming the following double Fourier series expansion for the mid-surface displacements u_0 and v_0 and the deflection w_0

$$
\begin{aligned}
u_0(x,y) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} U_{mn} \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \\
v_0(x,y) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} V_{mn} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \\
w_0(x,y) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}
\end{aligned} \tag{6.33}
$$

6.3.2 Midsurface strains and curvatures

The midsurface strains, obtained by substituting for the mid-surface displacements from (6.33) into (3.12), are

$$
\begin{aligned}
\varepsilon_x^0 &= \frac{\partial u_0}{\partial x} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} U_{mn}\left(-\frac{m\pi}{a}\right) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \\
\varepsilon_y^0 &= \frac{\partial v_0}{\partial y} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} V_{mn}\left(-\frac{n\pi}{b}\right) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \\
\gamma_{xy}^0 &= \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \left[U_{mn}\left(\frac{n\pi}{b}\right) + V_{mn}\left(\frac{m\pi}{a}\right)\right] \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b}
\end{aligned} \tag{6.34}
$$

The midsurface curvatures, obtained by substituting for the mid-surface displacements from (6.33) into (3.13), are

$$\begin{aligned}
\kappa_x &= -\frac{\partial^2 w_0}{\partial x^2} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{mn}\left(\frac{m\pi}{a}\right)^2 \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \\
\kappa_y &= -\frac{\partial^2 w_0}{\partial y^2} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{mn}\left(\frac{n\pi}{b}\right)^2 \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \\
\kappa_{xy} &= -2\frac{\partial^2 w_0}{\partial x \partial y} = -2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{mn}\left(\frac{mn\pi^2}{ab}\right) \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}
\end{aligned} \tag{6.35}$$

6.3.3 Force and moment resultants

The force resultants, obtained using (3.13), (6.34), (6.35), are

$$\begin{aligned}
N_x &= A_{11}\varepsilon_x^0 + A_{12}\varepsilon_y^0 + B_{11}\kappa_x + B_{12}\kappa_y \\
&= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[-A_{11}\left(\frac{m\pi}{a}\right)U_{mn} - A_{12}\left(\frac{n\pi}{b}\right)V_{mn} + \left(B_{11}\left(\frac{m\pi}{a}\right)^2 + B_{12}\left(\frac{n\pi}{b}\right)^2\right)W_{mn}\right]\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}
\end{aligned} \tag{6.36a}$$

$$\begin{aligned}
N_y &= A_{12}\varepsilon_x^0 + A_{22}\varepsilon_y^0 + B_{12}\kappa_x + B_{22}\kappa_y \\
&= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[-A_{12}\left(\frac{m\pi}{a}\right)U_{mn} - A_{22}\left(\frac{n\pi}{b}\right)V_{mn} + \left(B_{12}\left(\frac{m\pi}{a}\right)^2 + B_{22}\left(\frac{n\pi}{b}\right)^2\right)W_{mn}\right]\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}
\end{aligned} \tag{6.36b}$$

$$\begin{aligned}
N_{xy} &= A_{66}\gamma_{xy}^0 + B_{66}\kappa_{xy} \\
&= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\left(U_{mn}\left(\frac{n\pi}{b}\right) + V_{mn}\left(\frac{m\pi}{a}\right)\right)A_{66} - 2B_{66}\left(\frac{mn\pi^2}{ab}\right)W_{mn}\right]\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}
\end{aligned} \tag{6.36c}$$

The moment resultants, obtained using (3.24), (6.34), (6.35), are

$$\begin{aligned} M_x &= B_{11}\varepsilon_x^0 + B_{12}\varepsilon_y^0 + D_{11}\kappa_x + D_{12}\kappa_y \\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[-B_{11}\left(\frac{m\pi}{a}\right)U_{mn} - B_{12}\left(\frac{n\pi}{b}\right)V_{mn} + \left(D_{11}\left(\frac{m\pi}{a}\right)^2 + D_{12}\left(\frac{n\pi}{b}\right)^2\right)W_{mn}\right]\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \end{aligned} \tag{6.37a}$$

$$\begin{aligned} M_y &= B_{12}\varepsilon_x^0 + B_{22}\varepsilon_y^0 + D_{12}\kappa_x + D_{22}\kappa_y \\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[-B_{12}\left(\frac{m\pi}{a}\right)U_{mn} - B_{22}\left(\frac{n\pi}{b}\right)V_{mn} + \left(D_{12}\left(\frac{m\pi}{a}\right)^2 + D_{22}\left(\frac{n\pi}{b}\right)^2\right)W_{mn}\right]\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \end{aligned} \tag{6.37b}$$

$$\begin{aligned} M_{xy} &= B_{66}\gamma_{xy}^0 + D_{66}\kappa_{xy} \\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\left(U_{mn}\left(\frac{n\pi}{b}\right) + V_{mn}\left(\frac{m\pi}{a}\right)\right)B_{66} - 2D_{66}\left(\frac{mn\pi^2}{ab}\right)W_{mn}\right]\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \end{aligned} \tag{6.37c}$$

6.3.4 Equilibrium equations

Substituting the force resultants N_x and N_{xy} from (6.36a) and (6.36c) into the equilibrium equation (4.28a)

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \tag{6.38}$$

gives

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[-K_{11}U_{mn} - K_{12}V_{mn} - K_{13}W_{mn}\right]\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b} = 0 \tag{6.39}$$

which reduces to

$$K_{11}U_{mn} + K_{12}V_{mn} + K_{13}W_{mn} = 0 \tag{6.40}$$

where,

$$\begin{aligned} K_{11} &= A_{11}\left(\frac{m\pi}{a}\right)^2 + A_{66}\left(\frac{n\pi}{b}\right)^2 \\ K_{12} &= (A_{12} + A_{66})\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right) \\ K_{13} &= -B_{11}\left(\frac{m\pi}{a}\right)^3 - B_{12}\left(\frac{n\pi}{b}\right)^2\left(\frac{m\pi}{a}\right) - 2B_{66}\left(\frac{mn\pi^2}{ab}\right)\left(\frac{n\pi}{b}\right) \\ &= -B_{11}\left(\frac{m\pi}{a}\right)^3 - (B_{12} + 2B_{66})\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)^2 \end{aligned} \tag{6.41}$$

Substituting the force resultants N_{xy} and N_y from (6.36c) and (6.36b) into the equilibrium equation (4.28b)

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0 \tag{6.42}$$

gives

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \left[-K_{21}U_{mn} - K_{22}V_{mn} - K_{23}W_{mn}\right] \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} = 0 \tag{6.43}$$

which reduces to

$$K_{21}U_{mn} + K_{22}V_{mn} + K_{23}W_{mn} = 0 \tag{6.44}$$

where

$$\begin{aligned}
K_{21} &= (A_{12} + A_{66})\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)\\
K_{22} &= A_{66}\left(\frac{m\pi}{a}\right)^2 + A_{22}\left(\frac{n\pi}{b}\right)^2\\
K_{23} &= -2B_{66}\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right) - B_{12}\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right) - B_{22}\left(\frac{n\pi}{b}\right)^3\\
&= -(B_{12} + 2B_{66})\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right) - B_{22}\left(\frac{n\pi}{b}\right)^3
\end{aligned} \tag{6.45}$$

Substituting the moment resultants M_x, M_y and M_{xy} from (6.37) into the equilibrium equation (4.28c)

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + q\,(x, y) = 0 \tag{6.46}$$

gives

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \left[-K_{31}U_{mn} - K_{32}V_{mn} - K_{33}W_{mn} + q_{mn}\right] \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} = 0 \tag{6.47}$$

which reduces to

$$K_{31}U_{mn} + K_{32}V_{mn} + K_{33}W_{mn} = q_{mn} \tag{6.48}$$

where

$$
\begin{aligned}
K_{31} &= -B_{11}\left(\frac{m\pi}{a}\right)^3 - 2B_{66}\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)^2 - B_{12}\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)^2 \\
&= -B_{11}\left(\frac{m\pi}{a}\right)^3 - \left(B_{12}+2B_{66}\right)\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)^2 \\
K_{32} &= -B_{12}\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right) - 2B_{66}\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right) - B_{22}\left(\frac{n\pi}{b}\right)^3 \\
&= -\left(B_{12}+2B_{66}\right)\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right) - B_{22}\left(\frac{n\pi}{b}\right)^3 \\
K_{33} &= D_{11}\left(\frac{m\pi}{a}\right)^4 + D_{12}\left(\frac{n\pi}{b}\right)^2\left(\frac{m\pi}{a}\right)^2 + 4D_{66}\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 \\
&+ \left[D_{12}\left(\frac{m\pi}{a}\right)^2 + D_{22}\left(\frac{n\pi}{b}\right)^2\right]\left(\frac{n\pi}{b}\right)^2 \\
&= D_{11}\left(\frac{m\pi}{a}\right)^4 + 2\left(D_{12}+2D_{66}\right)\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + D_{22}\left(\frac{n\pi}{b}\right)^4
\end{aligned}
\tag{6.49}
$$

6.3.5 Solution for the displacement coefficients

Equations (6.40), (6.44) and (6.48) can be written in matrix form as:

$$
\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{12} & K_{22} & K_{23} \\ K_{13} & K_{23} & K_{33} \end{bmatrix}
\begin{Bmatrix} U_{mn} \\ V_{mn} \\ W_{mn} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ q_{mn} \end{Bmatrix}
\tag{6.50}
$$

where K_{ij} are defined in (6.41), (6.45) and (6.49) and $[K]$ is symmetric since $K_{ji} = K_{ij}$. The displacement coefficients U_{mn}, V_{mn} and W_{mn} can be obtained by solving equation (6.50) numerically for each combination of m and n. Once we have the displacement coefficients U_{mn}, V_{mn} and W_{mn}, the midsurface strains and curvatures at any location (x, y) can be determined using (6.34) and (6.35). In practice, We usually truncate the infinite series to a finite number of terms

$$
\begin{aligned}
u_0(x,y) &= \sum_{m=1}^{N_s}\sum_{n=1}^{N_s} U_{mn}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \\
v_0(x,y) &= \sum_{m=1}^{N_s}\sum_{n=1}^{N_s} V_{mn}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \\
w_0(x,y) &= \sum_{m=1}^{N_s}\sum_{n=1}^{N_s} W_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}
\end{aligned}
\tag{6.51}
$$

where N_s defines the number of terms retained in the series. As discussed earlier, it is necessary to check for the convergence of displacements and stresses to make sure that a sufficient number of terms has been used in the series solution.

6.4 Bending of antisymmetric angle-ply laminated plates

It is possible to develop a Navier solution for antisymemtric angle-ply laminates for certain boundary conditions. Consider an antisymmetric angle-ply laminate that is subjected to the following S3 boundary conditions on all four edges as shown in Fig. 6.6,

$$\begin{aligned} w_0 = 0\,,\ M_x = 0\,,\ u_0 = 0\,,\ N_{xy} = 0 \quad &\text{at} \quad x = 0,\ a \\ w_0 = 0\,,\ M_y = 0\,,\ N_{xy} = 0\,,\ v_0 = 0 \quad &\text{at} \quad y = 0,\ b \end{aligned} \tag{6.52}$$

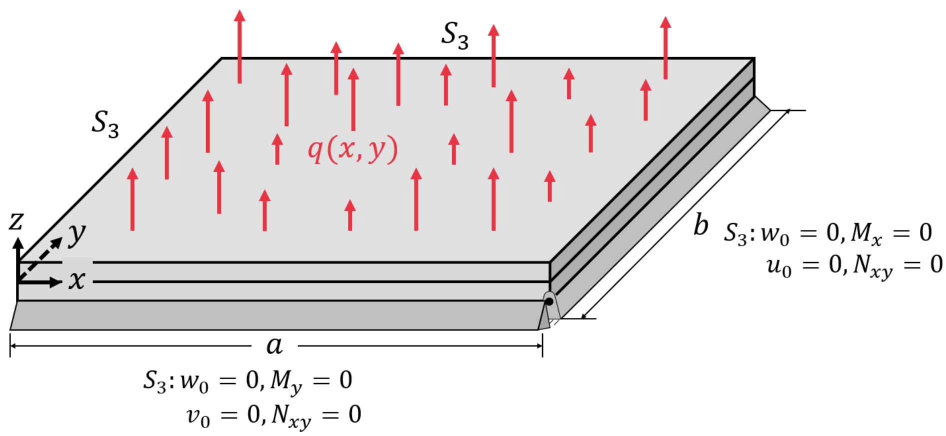


Figure 6.6: Simply supported laminated plate subjected S3 boundary conditions on all four edges

An antisymmetric angle-ply laminate has an even number of orthotropic layers with the principal material directions oriented at θ on one side and $-\theta$ on the other side of the midsurface, with $0° \le \theta \le 90°$. Examples of antisymmetric laminates include [−45/30/−15/15/−30/45] and [0/45/90/60/−60/−90/−45/0]. In the case of an antisymmetic angle-ply laminates, the following laminate rigidities are zero,

$$A_{16} = A_{26} = 0, \quad B_{11} = B_{22} = B_{12} = B_{66} = 0, \quad D_{16} = D_{26} = 0 \tag{6.53}$$

The Navier solutions for the midsurface displacements of antisymmetric angle-ply laminates is

$$\begin{aligned} u_0(x, y) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} U_{mn} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \\ v_0(x, y) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} V_{mn} \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \\ w_0(x, y) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \end{aligned} \tag{6.54}$$

We can use a similar procedure as before to solve for the displacement coefficients U_{mn}, V_{mn} and W_{mn}.

In the case of antisymmetric laminates we obtain the following system of equations

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{12} & K_{22} & K_{23} \\ K_{13} & K_{23} & K_{33} \end{bmatrix} \begin{Bmatrix} U_{mn} \\ V_{mn} \\ W_{mn} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ q_{mn} \end{Bmatrix} \tag{6.55}$$

where the coefficients K_{ij} in the square matrix are

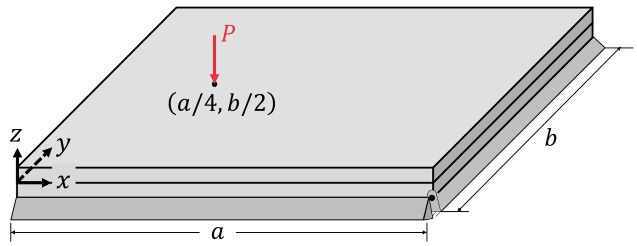
$$\begin{aligned} K_{11} &= A_{11}\left(\frac{m\pi}{a}\right)^2 + A_{66}\left(\frac{n\pi}{b}\right)^2 \\ K_{12} &= (A_{12} + A_{66})\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right) \\ K_{13} &= -3B_{16}\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right) - B_{26}\left(\frac{n\pi}{b}\right)^3 \\ K_{22} &= A_{22}\left(\frac{n\pi}{b}\right)^2 + A_{66}\left(\frac{m\pi}{a}\right)^2 \\ K_{23} &= -B_{16}\left(\frac{m\pi}{a}\right)^3 - 3B_{26}\left(\frac{m\pi}{a}\right)\left(\frac{n\pi}{b}\right)^2 \\ K_{33} &= D_{11}\left(\frac{m\pi}{a}\right)^4 + D_{22}\left(\frac{n\pi}{b}\right)^4 + 2\left(D_{12} + 2D_{66}\right)\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 \end{aligned} \tag{6.56}$$

Upon solving for the displacement coefficients U_{mn}, V_{mn} and W_{mn} from (6.55), the midsurface displacements can be obtained from (6.54). Subsequently, the mid-surface strains and curvatures at any location (x, y) can be determined using (3.12) and (3.13), respectively. Once the mid-surface strains and curvatures have been determined, the through-thickness variation of the strains and stresses can be evaluated using (3.16).

Exercises

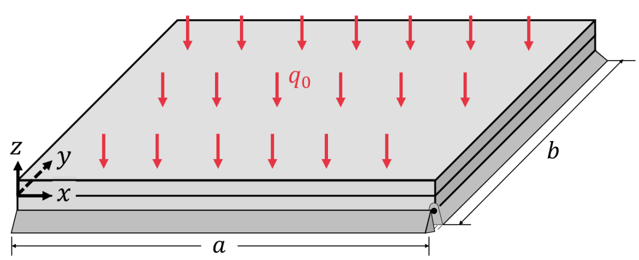
Use the representative properties in Sec. 1.10.1 for unidirectional carbon/epoxy composites and the representative properties in Sec. 1.10.2 for fabric-reinforced carbon/epoxy composites. Assume that the ply thickness h of the unidirectional and fabric-reinforced laminae are 0.2 mm and 0.4 mm, respectively. Use the Tsai-Wu theory for failure analysis.

6.1 Consider the bending of a $[0/90]_S$ carbon fiber-reinforced square plate with unidirectional plies. The length and width of the laminated plate are $a = b = 0.5$ m. All four edges of the plate are simply supported with S_1 boundary conditions. The plate is subjected to an off-center point load of magnitude $P = 1$ N acting downward at $x = a/4$ and $y = b/2$ as shown in the figure. Analyze the bending of the specially orthotropic plate using the Navier solution by truncating the infinite series to a finite series with summation ranging from 1 to N_s.



(a) Evaluate the deflection $w_0(a/2, b/2)$ at the center of the plate. Tabulate the deflection for $N_s = 5, 25$ and 50. Comment on the convergence of the deflection as N_s is increased.
(b) Plot the converged deflection $w_0(x, b/2)$ vs. x. Determine the maximum deflection $w_0^{\max}$. Does the maximum deflection occur at the point of application of the load?
(c) Plot the converged deflection $w_0(a/4, y)$ vs. y. Is the plot of the deflection in the y-direction qualitatively similar to the plot of the deflection in the x-direction?
(d) Evaluate the curvatures at the center of the plate and the normal stress $\sigma_x(a/2, b/2, -H/2)$ on the bottom surface. Tabulate the normal stress $\sigma_x(a/2, b/2, -H/2)$ for $N_s = 5, 25$ and 50. Comment on the convergence of the normal stress σ_x as compared to the convergence of the deflection w_0 as N_s is increased.
(e) Plot the through-thickness variation of the stress components σ_x, σ_1, σ_2 and the safety factor S_{fa} at the center of the plate and obtain the minimum safety factor S_{fa}^{min} at that location.

6.2 Consider the bending of a $[0_2/90_2]$ carbon fiber-reinforced rectangular plate with unidirectional plies. The length and width of the laminated plate are $a = 0.6$ m and $b = 0.4$ m, respectively. All four edges of the plate are simply supported with S_2 boundary conditions. The plate is subjected to a uniform distributed load of magnitude $q_0 = 5\ \mathrm{N/m^2}$ acting downward as shown in the figure. Analyze the bending of the unsymmetric cross-ply laminated plate using the Navier solution.



(a) Plot the deflection $w_0(x, b/2)$ vs. x. Determine the maximum deflection $w_0^{\max}$ at the center of the plate.
(b) Plot the in-plane displacement $u_0(x, b/2)$ vs. x. Determine the in-plane displacement $u_0(0, b/2)$ at the mid-point of the edge at $x = 0$.
(c) Evaluate the mid-surface strains and curvatures at the center of the plate.
(d) Plot the through-thickness variation of the stress components σ_x, σ_1, σ_2 and the safety factor S_{fa} at the center of the plate and obtain the minimum safety factor S_{fa}^{min} at that location.

Approximate Solutions for Bending 7

7.1 Principle of minimum total potential energy

Suppose an elastic body occupying the region V with boundary S is subjected to forces $\boldsymbol{F}$ and surface traction $\boldsymbol{f}$ on S_f as shown in Fig. 7.1. The elastic body is fixed on the boundary S_u. Let's consider a displacement field $\boldsymbol{u}(\boldsymbol{x})$ that satisfies the displacement boundary conditions $\boldsymbol{u} = \boldsymbol{0}$ on S_u. The corresponding strains and stresses are denoted as $\boldsymbol{\varepsilon}(\boldsymbol{x})$ and $\boldsymbol{\sigma}(\boldsymbol{x})$, respectively.

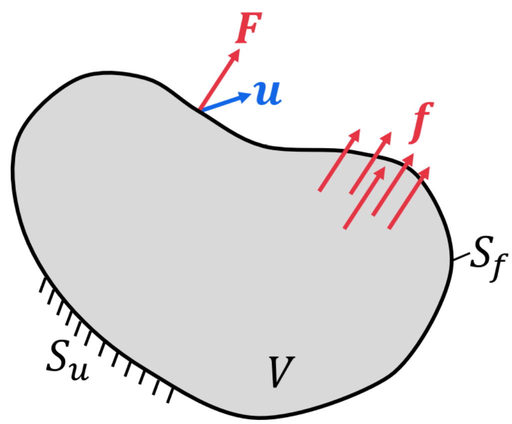


Figure 7.1: Elastic body under applied loads

The total internal strain energy U_s due to the deformation is obtained by integrating the strain energy density U over the volume V,

$$U_s(\boldsymbol{u}) = \int_V U dv = \frac{1}{2}\int_V \sigma_{ij}\varepsilon_{ij} dv \tag{7.1}$$

When a force $\boldsymbol{F}$ acting on elastic body displaces by an amount $\boldsymbol{u}$, it loses some potential to do additional work. Here its potential energy is defined as the negative of the product of the force and the displacement in the direction of the force. Hence, the potential energy Ω of the external loads is defined as

$$\Omega(\boldsymbol{u}) = -\boldsymbol{F}\cdot\boldsymbol{u} - \int_{S_f} \boldsymbol{f}\cdot\boldsymbol{u} da = -F_i u_i - \int_{S_f} f_i u_i da \tag{7.2}$$

Note that the reaction forces on the fixed boundary do not contribute to the potential energy of the external loads since the displacement $\boldsymbol{u} = \boldsymbol{0}$ on S_u.

The total potential energy Π of the system, which includes the elastic body and the external loads, is defined as the sum of the elastic strain energy U_s and the potential energy Ω of the external loads,

$$\Pi(\boldsymbol{u}) = U_s(\boldsymbol{u}) + \Omega(\boldsymbol{u}) \tag{7.3}$$

where $\boldsymbol{u}$ is an admissible displacement field that satisfies the displacement boundary conditions on S_u.

According to the principle of minimum total potential energy, *among all the possible admissible displacement fields, the actual/exact displacement field is that which minimizes the total potential energy of the system*. In other words, the body will deform in a manner that minimizes the total potential energy of the system. This can be stated as

$$\Pi(\boldsymbol{u}_a) \leq \Pi(\boldsymbol{u}) \tag{7.4}$$

where $\boldsymbol{u}$ is an arbitrary admissible displacement field and $\boldsymbol{u}_a$ is the actual/exact displacement field that satisfies elastic equilibrium.

EXAMPLE 7.1: Elastic bar under an axial load

Consider an elastic bar of length L and cross sectional area A that is subjected to an axial force F as shown in Fig. 7.2. We are interested in determining the elongation δ of the bar using the principle of minimum total potential energy.

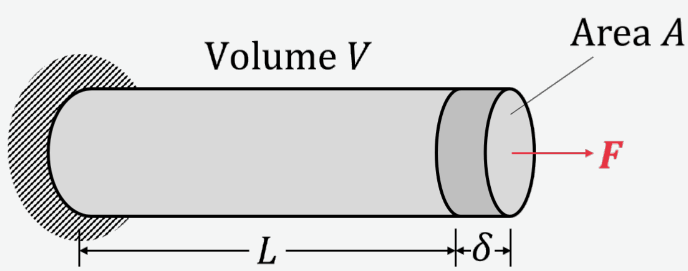


Figure 7.2: Axially loaded bar

Taking inspiration from the potential energy stored in a spring, we define the elastic potential/strain energy density stored in an axially loaded bar as

$$U = \frac{1}{2}E\varepsilon^2 \tag{7.5}$$

where E is a material property that represents the stiffness of the material and ε is the axial strain that characterizes the intensity of deformation

$$\varepsilon = \frac{\delta}{L} \tag{7.6}$$

The elastic strain energy density can be expressed in terms of the elongation δ by substituting for the strain ε from (7.6) into (7.5),

$$U = \frac{1}{2}\frac{E}{L^2}\delta^2 \tag{7.7}$$

The total elastic strain energy stored in the bar is obtained by integrating the strain energy density over the volume

$$U_s = \int_V U dv = \int_V \frac{E}{2L^2}\delta^2 dv = \frac{E\delta^2}{2L^{\cancel{2}}} A\cancel{L} \tag{7.8}$$

Thus, the total elastic strain energy stored in the bar is

$$U_s = \frac{EA}{2L}\delta^2 \tag{7.9}$$

When the force F displaces by a distance δ due to the elongation of the bar, it loses some potential to do additional work. Hence, its potential energy is defined as the negative of the product of the force and the corresponding displacement, i.e.,

$$\Omega = -F\delta \tag{7.10}$$

Thus the total potential energy of the system

$$\Pi = U_s + \Omega \tag{7.11}$$

can be expressed in terms of the elongation δ by substituting (7.9) and (7.10) into (7.11)

$$\Pi(\delta) = \frac{EA}{2L}\delta^2 - F\delta \tag{7.12}$$

The variation of the total potential energy of the system with respect to the elongation δ is shown in Fig. 7.3.

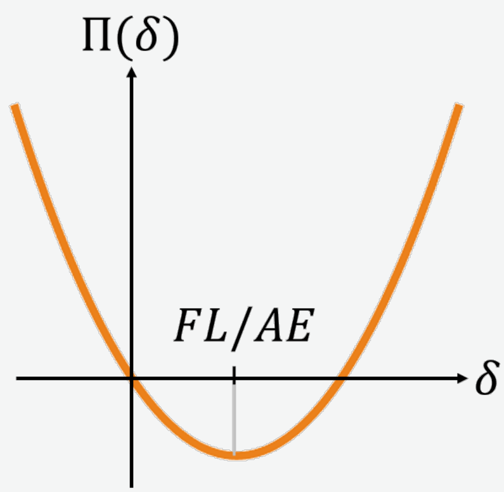


Figure 7.3: Variation of total potential energy Π with elongation δ

According to the principle of minimum total potential energy, at equilibrium

$$\frac{\partial \Pi(\delta)}{\partial \delta} = \frac{\cancel{2} E A \delta}{\cancel{2} L} - F = 0 \tag{7.13}$$

from which we obtain the actual elongation δ as

$$\delta = \frac{FL}{AE} \tag{7.14}$$

Note, that we obtained the elongation δ directly from the strain energy function (7.5) and the principle of minimum potential energy without using Hooke's law.

7.2 Total potential energy of a laminated rectangular plate

Consider a laminated rectangular plate that is subjected to arbitrary boundary conditions on its edges and a distributed load $q(x, y)$ as shown in Fig. 7.4. We are interested in determining the deflection and

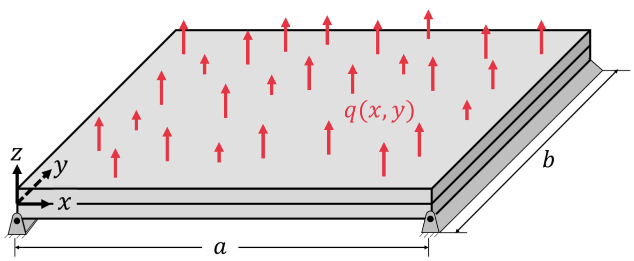


Figure 7.4: Laminated plate subjected to a distributed load

stresses in the laminated plate using the principle of minimum total potential energy. To do that, we need to first evaluate the strain energy stored in the laminated plate due to the deformation and the potential energy of external loads.

7.2.1 Strain energy of a laminated plate

When a laminated composite plate is subjected to loads, the resulting strain energy can be obtained by integrating the strain energy density over the volume of the plate,

$$\begin{aligned} U_s &= \frac{1}{2}\int_V \left(\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \cancelto{0}{\sigma_z}\,\varepsilon_z + \cancelto{0}{\tau_{yz}}\,\gamma_{yz} + \cancelto{0}{\tau_{xz}}\,\gamma_{xz} + \tau_{xy}\gamma_{xy}\right) dv \\ &= \frac{1}{2}\int_V \left(\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \tau_{xy}\gamma_{xy}\right) dv \end{aligned} \tag{7.15}$$

where we have used the plane stress assumption. Recall the stress-strain relations in the global coordinate system

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{7.16}$$

where $\overline{Q}_{ij}$ are the off-axis stiffnesses. Substituting for the stresses from (7.16) into (7.15), we obtain the total strain energy of the laminated plate in terms of the strains

$$U_s = \frac{1}{2}\int_V \left(\overline{Q}_{11}\varepsilon_x^2 + 2\overline{Q}_{12}\varepsilon_x\varepsilon_y + 2\overline{Q}_{16}\varepsilon_x\gamma_{xy} + 2\overline{Q}_{26}\varepsilon_y\gamma_{xy} + \overline{Q}_{22}\varepsilon_y^2 + \overline{Q}_{66}\gamma_{xy}^2\right) dv \tag{7.17}$$

Next, we invoke the Kirchhoff hypothesis

$$\begin{aligned} \varepsilon_x &= \varepsilon_x^0 + z\kappa_x \\ \varepsilon_y &= \varepsilon_y^0 + z\kappa_y \\ \gamma_{xy} &= \gamma_{xy}^0 + z\kappa_{xy} \end{aligned} \tag{7.18}$$

and substitute for the strains from (7.18) into (7.17) to obtain the total strain energy in terms of the mid-surface strains and curvatures,

$$\begin{aligned} U_s = \frac{1}{2}\int_V \Big\{ & \overline{Q}_{11}\left[\left(\varepsilon_x^0\right)^2 + 2z\varepsilon_x^0\kappa_x + z^2\kappa_x^2\right] + 2\overline{Q}_{12}\left[\varepsilon_x^0\varepsilon_y^0 + z\varepsilon_x^0\kappa_y + z\varepsilon_y^0\kappa_x + z^2\kappa_x\kappa_y\right] \\ & + 2\overline{Q}_{16}\left[\varepsilon_x^0\gamma_{xy}^0 + z\varepsilon_x^0\kappa_{xy} + z\gamma_{xy}^0\kappa_x + z^2\kappa_x\kappa_{xy}\right] + 2\overline{Q}_{26}\left[\varepsilon_y^0\gamma_{xy}^0 + z\varepsilon_y^0\kappa_{xy} + z\gamma_{xy}^0\kappa_y + z^2\kappa_y\kappa_{xy}\right] \\ & + \overline{Q}_{22}\left[\left(\varepsilon_y^0\right)^2 + 2z\varepsilon_y^0\kappa_y + z^2\kappa_y^2\right] + \overline{Q}_{66}\left[\left(\gamma_{xy}^0\right)^2 + 2z\gamma_{xy}^0\kappa_{xy} + z^2\kappa_{xy}^2\right]\Big\} dv \end{aligned} \tag{7.19}$$

The volume integral in (7.19) is performed by first integrating through the thickness of the laminate. This gives the total strain energy in terms of the laminate rigidities, mid-surface strains and curvatures

$$\begin{aligned} U_s = \frac{1}{2}\int_A \Big\{ & A_{11}\left(\varepsilon_x^0\right)^2 + 2A_{12}\varepsilon_x^0\varepsilon_y^0 + A_{22}\left(\varepsilon_y^0\right)^2 + 2\left(A_{16}\varepsilon_x^0 + A_{26}\varepsilon_y^0\right)\gamma_{xy}^0 + A_{66}\left(\gamma_{xy}^0\right)^2 + 2B_{11}\varepsilon_x^0\kappa_x \\ & + 2B_{12}\left(\varepsilon_x^0\kappa_y + \varepsilon_y^0\kappa_x\right) + 2B_{16}\left(\varepsilon_x^0\kappa_{xy} + \gamma_{xy}^0\kappa_x\right) + 2B_{26}\left(\varepsilon_y^0\kappa_{xy} + \gamma_{xy}^0\kappa_y\right) + 2B_{22}\varepsilon_y^0\kappa_y \\ & + 2B_{66}\gamma_{xy}^0\kappa_{xy} + D_{11}\kappa_x^2 + 2D_{12}\kappa_x\kappa_y + 2\left(D_{16}\kappa_x + D_{26}\kappa_y\right)\kappa_{xy} + D_{22}\kappa_y^2 + D_{66}\kappa_{xy}^2\Big\} da \end{aligned} \tag{7.20}$$

This is the most general expression for the total strain energy in a laminated plate due deformation.

In the case of symmetric laminates in pure bending, the mid-surface strains $\varepsilon_x^0 = \varepsilon_y^0 = \gamma_{xy}^0 = 0$ since the laminate rigidities $B_{ij} = 0$. In this case, the total strain energy in (7.20) reduces to

$$\begin{aligned}
U_s &= \frac{1}{2}\int_A \left\{D_{11}\kappa_x^2 + 2D_{12}\kappa_x\kappa_y + D_{22}\kappa_y^2 + 2\left(D_{16}\kappa_x + D_{26}\kappa_y\right)\kappa_{xy} + D_{66}\kappa_{xy}^2\right\} da \\
&= \frac{1}{2}\int_0^a\int_0^b \left[D_{11}\left(\frac{\partial^2 w_0}{\partial x^2}\right)^2 + 2D_{12}\frac{\partial^2 w_0}{\partial x^2}\frac{\partial^2 w_0}{\partial y^2} + D_{22}\left(\frac{\partial^2 w_0}{\partial y^2}\right)^2\right. \\
&\qquad \left. + 4\left(D_{16}\frac{\partial^2 w_0}{\partial x^2} + D_{26}\frac{\partial^2 w_0}{\partial y^2}\right)\frac{\partial^2 w_0}{\partial x \partial y} + 4D_{66}\left(\frac{\partial^2 w_0}{\partial x \partial y}\right)^2\right] dxdy
\end{aligned} \tag{7.21}$$

7.2.2 Potential energy of external loads

When a laminated plate is subjected to a distributed load, the distributed load q displaces an area element da by an amount w_0 in the z-direction. Thus the distributed load loses some potential to do additional work on the laminated plate. Therefore, the potential energy of the distributed load is defined as

$$\Omega = -\int_A q w_0 da = -\int_0^a\int_0^b q(x,y)w_0(x,y)dxdy \tag{7.22}$$

7.3 Approximate solution using the Ritz method

The Ritz method is a convenient technique for obtaining approximate solutions to boundary value problems. In the Ritz method, the solution is sought in the form

$$\begin{aligned}
u_0(x,y) &= \sum_{m=1}^{M_1}\sum_{n=1}^{N_1} U_{mn}u_{mn}(x,y) \\
v_0(x,y) &= \sum_{m=1}^{M_2}\sum_{n=1}^{N_2} V_{mn}v_{mn}(x,y) \\
w_0(x,y) &= \sum_{m=1}^{M_3}\sum_{n=1}^{N_3} W_{mn}w_{mn}(x,y)
\end{aligned} \tag{7.23}$$

where U_{mn}, V_{mn} and W_{mn} are undetermined coefficients. The functions $u_{mn}(x,y)$, $v_{mn}(x,y)$ and $w_{mn}(x,y)$ are chosen to qualitatively resemble the anticipated deformation of the plate. The essential geometric boundary conditions involving the displacements or slopes must be satisfied by the chosen functions.

When the midsurface strains and curvatures corresponding to the chosen displacements (7.23) are substituted into (7.20), the resulting total potential energy Π is a function of U_{mn}, V_{mn} and W_{mn}, i.e.,

$$\Pi = \Pi\left(U_{mn}, V_{mn}, W_{mn}\right) \tag{7.24}$$

The principle of minimum total potential energy states that,

$$\Pi\left(U_{mn}, V_{mn}, W_{mn}\right) = \text{stationary value} \tag{7.25}$$

This condition yields the following system of equations,

$$\begin{aligned}
\frac{\partial \Pi}{\partial U_{mn}} &= 0 \qquad \text{where} \qquad m = 1, 2, \ldots M_1\ ; \ \ n = 1, 2, \ldots N_1 \\
\frac{\partial \Pi}{\partial V_{mn}} &= 0 \qquad \text{where} \qquad m = 1, 2, \ldots M_2\ ; \ \ n = 1, 2, \ldots N_2 \\
\frac{\partial \Pi}{\partial W_{mn}} &= 0 \qquad \text{where} \qquad m = 1, 2, \ldots M_3\ ; \ \ n = 1, 2, \ldots N_3
\end{aligned} \tag{7.26}$$

In the formulation presented here, the total potential energy of the system Π is always a quadratic function of the undetermined coefficients. Thus the conditions above, in (7.26), are a $\sum_{i=1}^{3} M_i \times N_i$ set of linear simultaneous equations for the unknown coefficients U_{mn}, V_{mn} and W_{mn}. Upon solving the simultaneous equations for the unknown coefficients, we can evaluate the mid-surface displacements using (7.23). Subsequently, we can determine the mid-surface strains, curvatures, and the strains and stresses at any location within the laminated plate.

7.4 Bending of specially orthotropic rectangular plates

Consider a specially orthotropic laminated rectangular plate that is supported by arbitrary boundary conditions on its four edges as shown in Fig. 7.5. The plate is subjected to a uniform distributed load of

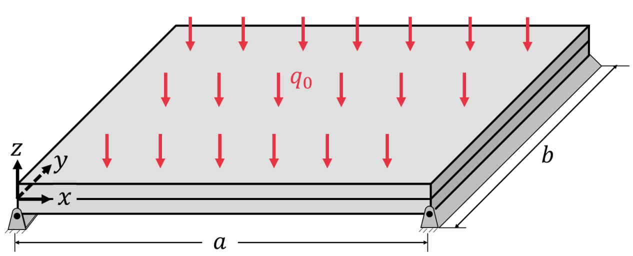


Figure 7.5: Laminated plate subjected to a uniform distributed load

magnitude q_0 acting downward, i.e.,

$$q\left(x, y\right) = -q_0 \tag{7.27}$$

Recall that the following laminate rigidities are zero for a specially orthotropic laminate

$$A_{16} = A_{26} = 0, \quad [B] = [0], \quad D_{16} = D_{26} = 0 \tag{7.28}$$

Since the bending-extension coupling rigidities B_{ij} are zero, bending of the laminate will not induce in-plane mid-surface strains and displacements. Therefore, it follows that

$$u_0 = v_0 = 0, \quad w_0 = w_0(x, y) \tag{7.29}$$

We use the Ritz solution to obtain an approximate solution for the laminated plate. The total strain energy, obtained by setting $D_{16} = D_{26} = 0$ in (7.21), is

$$U_s = \frac{1}{2}\int_0^a \int_0^b \left[D_{11}\left(\frac{\partial^2 w_0}{\partial x^2}\right)^2 + 2D_{12}\frac{\partial^2 w_0}{\partial x^2}\frac{\partial^2 w_0}{\partial y^2} + D_{22}\left(\frac{\partial^2 w_0}{\partial y^2}\right)^2 + 4D_{66}\left(\frac{\partial^2 w_0}{\partial x \partial y}\right)^2 \right] dxdy \tag{7.30}$$

The potential energy of the external loads is given by

$$\Omega = -\int_0^a \int_0^b q(x, y)\, w_0(x, y)\, dxdy = q_0 \int_0^a \int_0^b w_0(x, y)\, dxdy \tag{7.31}$$

The total potential energy of the system follows from (7.30) and (7.31) as

$$\begin{aligned} \Pi &= U_s + \Omega \\ &= \frac{1}{2}\int_0^a \int_0^b \left[D_{11}\left(\frac{\partial^2 w_0}{\partial x^2}\right)^2 + 2D_{12}\frac{\partial^2 w_0}{\partial x^2}\frac{\partial^2 w_0}{\partial y^2} + D_{22}\left(\frac{\partial^2 w_0}{\partial y^2}\right)^2 + 4D_{66}\left(\frac{\partial^2 w_0}{\partial x \partial y}\right)^2 + 2q_0 w_0 \right] dxdy \end{aligned} \tag{7.32}$$

In order to find an approximate solution for the plate deflection, we consider the following finite series in variable separable form,

$$w_0(x, y) = \sum_{m=1}^{M}\sum_{n=1}^{N} W_{mn} X_m(x)\, Y_n(y) \tag{7.33}$$

where $X_m(x)$ and $Y_n(y)$ are functions that are chosen to satisfy the essential boundary conditions and W_{mn} are unknown coefficients. By the principle of minimum total potential energy,

$$\frac{\partial \Pi}{\partial W_{mn}} = 0, \qquad m = 1, 2, \ldots M; \quad n = 1, 2, \ldots N \tag{7.34}$$

The relations in (7.34) give $M \times N$ equations for the $M \times N$ unknowns W_{mn}.

Differentiating (7.33) for the curvatures we obtain:

$$\frac{\partial^2 w_0}{\partial x^2} = \sum_{m=1}^{M}\sum_{n=1}^{N} W_{mn} X_m'' Y_n$$
$$\frac{\partial^2 w_0}{\partial y^2} = \sum_{m=1}^{M}\sum_{n=1}^{N} W_{mn} X_m Y_n'' \tag{7.35}$$

Now differentiating (7.32) summand by summand:

$$\begin{aligned}
\frac{\partial}{\partial W_{mn}}\left[\frac{1}{2}D_{11}\left(\frac{\partial^2 w_0}{\partial x^2}\right)^2\right] &= D_{11}\left(\frac{\partial^2 w_0}{\partial x^2}\right)\frac{\partial}{\partial W_{mn}}\left(\frac{\partial^2 w_0}{\partial x^2}\right)\\
&= D_{11}\left(\frac{\partial^2 w_0}{\partial x^2}\right)\frac{\partial}{\partial W_{mn}}\left[\sum_{i=1}^{M}\sum_{j=1}^{N} W_{ij} X_i'' Y_j\right]\\
&= D_{11}\left(\sum_{i=1}^{M}\sum_{j=1}^{N} W_{ij} X_i'' Y_j\right) X_m'' Y_n\\
&= \sum_{i=1}^{M}\sum_{j=1}^{N}\left[D_{11}\left(X_i'' X_m''\right)\left(Y_j Y_n\right) W_{ij}\right]
\end{aligned} \tag{7.36a}$$

Similarly,

$$\begin{aligned}
\frac{\partial}{\partial W_{mn}}\left[\frac{1}{2}\cdot 2D_{12}\frac{\partial^2 w_0}{\partial x^2}\frac{\partial^2 w_0}{\partial y^2}\right] &= D_{12}\left[\frac{\partial}{\partial W_{mn}}\left(\frac{\partial^2 w_0}{\partial x^2}\right)\frac{\partial^2 w_0}{\partial y^2} + \frac{\partial^2 w_0}{\partial x^2}\frac{\partial}{\partial W_{mn}}\left(\frac{\partial^2 w_0}{\partial y^2}\right)\right]\\
&= D_{12}\left[X_m'' Y_n \sum_{i=1}^{M}\sum_{j=1}^{N} W_{ij} X_i Y_j'' + X_m Y_n'' \sum_{i=1}^{M}\sum_{j=1}^{N} W_{ij} X_i'' Y_j\right]\\
&= \sum_{i=1}^{M}\sum_{j=1}^{N} D_{12}\left[\left(X_m'' X_i\right)\left(Y_n Y_j''\right) + \left(X_i'' X_m\right)\left(Y_j Y_n''\right)\right] W_{ij}
\end{aligned} \tag{7.36b}$$

$$\frac{\partial}{\partial W_{mn}}\left[\frac{1}{2}D_{22}\left(\frac{\partial^2 w_0}{\partial y^2}\right)^2\right] = \sum_{i=1}^{M}\sum_{j=1}^{N} D_{22}\left(X_i X_m\right)\left(Y_j'' Y_n''\right) W_{ij} \tag{7.36c}$$

$$\frac{\partial}{\partial W_{mn}}\left[\frac{1}{2}\cdot 4D_{66}\left(\frac{\partial^2 w_0}{\partial x \partial y}\right)^2\right] = 4\sum_{i=1}^{M}\sum_{j=1}^{N} D_{66}\left(X_i' X_m'\right)\left(Y_j' Y_n'\right) W_{ij} \tag{7.36d}$$

$$\frac{\partial}{\partial W_{mn}}\left[\frac{1}{2}\left(2q_0 w_0\right)\right] = q_0\frac{\partial}{\partial W_{mn}}\left[\sum_{i=1}^{M}\sum_{j=1}^{N} W_{ij} X_i Y_j\right] = q_0 X_m Y_n \tag{7.36e}$$

Substitution of (7.36) and (7.32) into the principle of minimum total potential energy (7.34) gives

$$\sum_{i=1}^{M}\sum_{j=1}^{N}\left\{D_{11}\left(\int_0^a X_i'' X_m'' dx\right)\left(\int_0^b Y_j Y_n dy\right)+D_{22}\left(\int_0^a X_i X_m dx\right)\left(\int_0^b Y_j'' Y_n'' dy\right)+\right.$$
$$D_{12}\left[\left(\int_0^a X_m'' X_i dx\right)\left(\int_0^b Y_n Y_j'' dy\right)+\left(\int_0^a X_i'' X_m dx\right)\left(\int_0^b Y_j Y_n'' dy\right)\right]+ \tag{7.37}$$
$$\left.4D_{66}\left(\int_0^a X_i' X_m' dx\right)\left(\int_0^b Y_j' Y_n' dy\right)\right\}W_{ij}=-q_0\left(\int_0^a X_m dx\right)\left(\int_0^b Y_n dy\right)$$

for $m = 1, 2, \ldots M$ and $n = 1, 2, \ldots N$. This yields a linear system of equations for the unknown coefficients W_{ij} where $i = 1, \ldots M$ and $j = 1, \ldots N$.

EXAMPLE 7.2: Bending of a specially orthotropic clamped rectangular plate

Consider a specially orthotropic rectangular plate of length a and width b that is clamped on all four edges and subjected to a uniform distributed load of magnitude q_0 as shown in Fig. 7.38. We use the Ritz method to obtain an approximate solution for the deflection.

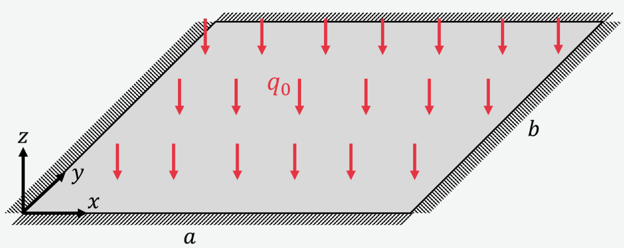


Figure 7.6: Clamped rectangular plate subjected to a uniform distributed load

The boundary conditions for the clamped edges of the plate are

$$\begin{aligned} w_0 &= \frac{\partial w_0}{\partial x} = 0 \qquad \text{at} \quad x = 0, a \\ w_0 &= \frac{\partial w_0}{\partial y} = 0 \qquad \text{at} \quad y = 0, b \end{aligned} \tag{7.38}$$

We choose a one term solution for w_0 in variable separable form

$$w_0(x, y) = W_{11} X_1(x) Y_1(y) \tag{7.39}$$

where the functions $X_1(x)$ and $Y_1(y)$ need to satisfy the essential boundary conditions at the four edges. In the case of a clamped rectangular plate, the chosen functions should satisfy the displacement and slope boundary conditions at the clamped edges. For example, we can choose a one term solution of the form,

$$X_1(x) = \left(\frac{x}{a}\right)^2 \left(1 - \frac{x}{a}\right)^2$$
$$Y_1(y) = \left(\frac{y}{b}\right)^2 \left(1 - \frac{y}{b}\right)^2 \tag{7.40}$$

which satisfies the boundary conditions (7.38) at $x = 0, a$ and $y = 0, b$.

In the case of a one-term solution, (7.37) reduces to

$$\begin{aligned} &\Big\{D_{11}\left(\int_0^a X_1''X_1''dx\right)\left(\int_0^b Y_1Y_1dy\right) + D_{22}\left(\int_0^a X_1X_1dx\right)\left(\int_0^b Y_1''Y_1''dy\right) + \\ &D_{12}\left[\left(\int_0^a X_1''X_1dx\right)\left(\int_0^b Y_1Y_1''dy\right) + \left(\int_0^a X_1''X_1dx\right)\left(\int_0^b Y_1Y_1''dy\right)\right] + \\ &4D_{66}\left(\int_0^a X_1'X_1'dx\right)\left(\int_0^b Y_1'Y_1'dy\right)\Big\} W_{11} = -q_0\left(\int_0^a X_1dx\right)\left(\int_0^b Y_1dy\right) \end{aligned} \tag{7.41}$$

Given the solution form (7.40), the integrals involving $X_1(x)$ in (7.41) can be evaluated analytical to obtain

$$\begin{aligned} &\int_0^a X_1(x)\,dx = \frac{a}{30}, \quad \int_0^a X_1(x)\,X_1(x)\,dx = \frac{a}{630}, \quad \int_0^a X_1'(x)\,X_1'(x)\,dx = \frac{2}{105a} \\ &\int_0^a X_1(x)\,X_1''(x)\,dx = -\frac{2}{105a}, \quad \int_0^a X_1''(x)\,X_1''(x)\,dx = \frac{4}{5a^3} \end{aligned} \tag{7.42}$$

Similarly, integrals involving $Y_1(y)$ in (7.41) are evaluated analytical to obtain

$$\begin{aligned} &\int_0^b Y_1(y)\,dy = \frac{b}{30}, \quad \int_0^b Y_1(y)\,Y_1(y)\,dy = \frac{b}{630}, \quad \int_0^b Y_1'(y)\,Y_1'(y)\,dy = \frac{2}{105b} \\ &\int_0^b Y_1(y)\,Y_1''(y)\,dy = -\frac{2}{105b}, \quad \int_0^b Y_1''(y)\,Y_1''(y)\,dy = \frac{4}{5b^3} \end{aligned} \tag{7.43}$$

Substituting of the integrals from (7.42) and (7.43) into (7.41) gives,

$$\begin{aligned} &\Big\{D_{11}\left(\frac{4}{5a^3}\right)\left(\frac{b}{630}\right) + D_{12}\left[\left(\frac{-2}{105a}\right)\left(\frac{-2}{105b}\right) + \left(\frac{-2}{105a}\right)\left(\frac{-2}{105b}\right)\right] + \\ &D_{22}\left(\frac{a}{630}\right)\left(\frac{4}{5b^3}\right) + 4D_{66}\left(\frac{2}{105a}\right)\left(\frac{2}{105b}\right)\Big\} W_{11} = -q_0\left(\frac{a}{30}\right)\left(\frac{b}{30}\right) \end{aligned} \tag{7.44}$$

Next, we multiply all the terms in (7.44) by $105^2/2ab$ to obtain an equation for W_{11}

$$\left[\frac{7}{a^4}D_{11} + \frac{4}{a^2b^2}\left(D_{12} + 2D_{66}\right) + \frac{7}{b^4}D_{22}\right] W_{11} = -\frac{49}{8}q_0 \tag{7.45}$$

Solving for W_{11}, we obtain

$$W_{11} = -\frac{49}{8} \cdot \frac{q_0}{\frac{7}{a^4}D_{11} + \frac{4}{a^2b^2}\left(D_{12} + 2D_{66}\right) + \frac{7}{b^4}D_{22}} \tag{7.46}$$

Substituting for W_{11} from (7.46) and $X_1(x)$ and $Y_1(y)$ from (7.40) into (7.39) gives the following form for the mid-surface displacement $w_0\left(x, y\right)$,

$$w_0\left(x, y\right) = -\frac{49}{8} \cdot \frac{q_0\left[\left(\frac{x}{a}\right)\left(1 - \frac{x}{a}\right)\right]^2\left[\left(\frac{y}{b}\right)\left(1 - \frac{y}{b}\right)\right]^2}{\frac{7}{a^4}D_{11} + \frac{4}{a^2b^2}\left(D_{12} + 2D_{66}\right) + \frac{7}{b^4}D_{22}} \tag{7.47}$$

The numerator and denominator of (7.47) can be multiplied by a^4 and expressed in the alternate form

$$w_0\left(x, y\right) = -\frac{49}{8} \cdot \frac{q_0a^4\left[\left(\frac{x}{a}\right)\left(1 - \frac{x}{a}\right)\right]^2\left[\left(\frac{y}{b}\right)\left(1 - \frac{y}{b}\right)\right]^2}{7D_{11} + 4\left(D_{12} + 2D_{66}\right)s^2 + 7D_{22}s^4} \tag{7.48}$$

where $s = a/b$ is the aspect ratio of the plate.

The maximum deflection, which occurs at the center where $x = a/2$ and $y = b/2$, can be determined from (7.48),

$$w_0^{max} = -0.003418 \cdot \frac{q_0a^4}{D_{11} + 0.5714\left(D_{12} + 2D_{66}\right)s^2 + D_{22}s^4} \tag{7.49}$$

In the case of a square isotropic plate ($s = a/b = 1$, $D_{11} = D_{22} = D_{12} + 2D_{66} = D$), the maximum deflection from (7.49) based on a one-term Ritz solution is,

$$w_0^{max} = -0.00133 \cdot \frac{q_0a^4}{D} \tag{7.50}$$

The "exact" solution for an isotropic plate obtained using a large number of terms in the series [5] is,

$$w_0^{max} = -0.00126 \cdot \frac{q_0a^4}{D} \tag{7.51}$$

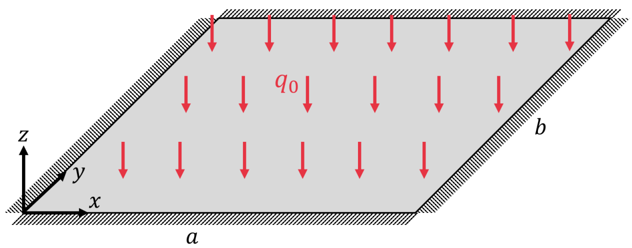
Thus, the error in the one-term polynomial for a square isotropic plate is 5.6%. It is possible to reduce the error by introducing more terms in the Ritz solution for the deflection $w_0\left(x, y\right)$.

Exercises

Use the representative properties in Sec. 1.10.1 for unidirectional carbon/epoxy composites and the representative properties in Sec. 1.10.2 for fabric-reinforced carbon/epoxy composites. Assume

that the ply thickness h of the unidirectional and fabric-reinforced laminae are 0.2 mm and 0.4 mm, respectively. Use the Tsai-Wu theory for failure analysis.

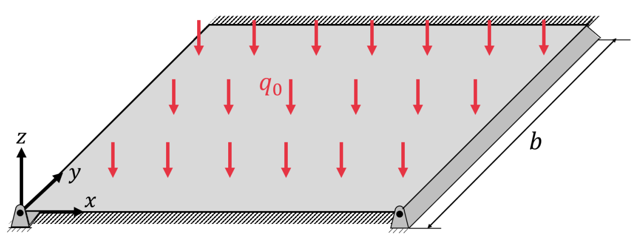
7.1 Consider the bending of a $[0/90]_S$ carbon fiber-reinforced rectangular plate with unidirectional plies. The length and width of the laminated plate are $a = 0.6$ m and $b = 0.4$ m, respectively. All four edges of the plate are clamped. The plate is subjected to a uniform distributed load of magnitude $q_0 = 10$ N/m^2 acting downward.



Analyze the bending of the laminated plate using the one-term Ritz solution for $w_0(x, y)$.

(a) Determine the maximum deflection $w_0^{\max}$ at the center of the plate
(b) Plot the deflection $w_0(x, b/2)$ vs. x
(c) Evaluate the curvatures κ_x, κ_y and κ_{xy} at a point with coordinates $(a/4, b/4)$
(d) Plot the through-thickness variation of the stress components σ_x, σ_1, σ_2 and the safety factor S_{fa} at the point with coordinates $(a/4, b/4)$ and obtain the minimum safety factor S_{fa}^{min} at that location.

7.2 Consider the bending of a $[0/90]_S$ carbon fiber-reinforced square plate with unidirectional plies. The length and width of the laminated plate are $a = b = 0.5$ m. The edges $x = 0$ and a are simply supported with S_1 boundary conditions. The other two edges, namely $y = 0$ and b, are clamped. The plate is subjected to a uniform distributed load of magnitude $q_0 = 10$ N/m^2 acting downward.



Analyze the bending of the laminated plate using a one-term Ritz solution of the form

$$w_0(x, y) = W_{11}\left(\frac{x}{a}\right)\left(1 - \frac{x}{a}\right)\left(\frac{y}{b}\right)^2\left(1 - \frac{y}{b}\right)^2$$

(a) Determine the maximum deflection $w_0^{\max}$ at the center of the plate

(b) Plot the deflection $w_0(x, b/2)$ vs. x

(c) Plot the deflection $w_0(a/2, y)$ vs. y

(d) Evaluate the curvatures κ_x, κ_y and κ_{xy} at the center of the plate. Do the relative magnitudes of the curvatures make sense?

(e) Plot the through-thickness variation of the stress components σ_x, σ_1, σ_2 and the safety factor S_{fa} at the center of the plate and obtain the minimum safety factor S_{fa}^{min} at that location.

8 Vibration of Laminated Plates

In this chapter, we will discuss the vibration of laminated composite plates.

8.1 Vibration of laminated plates

Let's consider laminated composite plates of uniform density wherein all laminae have the same density ρ. In that case, the areal mass I_0 and rotary inertia I_2 of the laminated plate are

$$I_0 = \rho H, \qquad I_2 = \frac{\rho H^3}{12} \tag{8.1}$$

where H is the thickness of the laminate. As previously demonstrated in Sec. 4.2.2, the density integral $I_1 = 0$. The equations of motion (4.6), (4.15) and (4.27) reduce to

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = I_0 \frac{\partial^2 u_0}{\partial t^2} \tag{8.2a}$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = I_0 \frac{\partial^2 v_0}{\partial t^2} \tag{8.2b}$$

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + q\left(x, y, t\right) = I_0 \frac{\partial^2 w_0}{\partial t^2} - I_2 \frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial^2 w_0}{\partial y^2}\right) \tag{8.2c}$$

where $q(x, y, t)$ is a distributed load.

8.1.1 Forced vibration

In the case of forced vibration, the distributed load is a function of time,

$$q = q\left(x, y, t\right) \tag{8.3}$$

For example, in the case of harmonic excitation,

$$q = q\left(x, y\right) \sin\left(\omega_f t\right) \tag{8.4}$$

where ω_f is the angular frequency of the applied force (forcing frequency).

8.1.2 Free vibration

In the case of free vibration, the plate is set into motion by initial conditions. The are no applied loads acting on the plate, i.e., $q(x, y, t) = 0$. The displacements can be written in the form

$$\begin{Bmatrix} u_0 \\ v_0 \\ w_0 \end{Bmatrix} = \begin{Bmatrix} U_0(x, y) \\ V_0(x, y) \\ W_0(x, y) \end{Bmatrix} \sin(\omega t + \phi) \tag{8.5}$$

where ω is a natural frequency that is a characteristic property of the system and is independent of the initial deflection or velocity of the plate. Alternatively, the harmonic variation of the displacements can be represented as follows

$$\begin{Bmatrix} u_0 \\ v_0 \\ w_0 \end{Bmatrix} = \begin{Bmatrix} U_0(x, y) \\ V_0(x, y) \\ W_0(x, y) \end{Bmatrix} e^{i\omega t} \tag{8.6}$$

8.1.3 Cylindrical bending vibration

In the case of cylindrical bending, $\frac{\partial(\cdot)}{\partial y} = 0$ and the equations of motion (8.2) reduce to

$$\frac{\partial N_x}{\partial x} = I_0 \frac{\partial^2 u_0}{\partial t^2} \tag{8.7a}$$

$$\frac{\partial N_{xy}}{\partial x} = I_0 \frac{\partial^2 v_0}{\partial t^2} \tag{8.7b}$$

$$\frac{\partial^2 M_x}{\partial x^2} + q(x, t) = I_0 \frac{\partial^2 w_0}{\partial t^2} - I_2 \frac{\partial^2}{\partial t^2} \left(\frac{\partial^2 w_0}{\partial x^2} \right) \tag{8.7c}$$

In the case of free vibration, the distributed load $q(x, t) = 0$. The displacements exhibit a harmonic variation in time which can be expressed as follows,

$$\begin{Bmatrix} u_0 \\ v_0 \\ w_0 \end{Bmatrix} = \begin{Bmatrix} U_0(x) \\ V_0(x) \\ W_0(x) \end{Bmatrix} e^{i\omega t} \tag{8.8}$$

where ω is a natural frequency.

8.2 Free vibration of simply supported laminates in cylindrical bending

In this section, we consider the free vibration of a simply supported cross-ply laminate of width a that is subjected to S_2 boundary conditions at $x = 0$ and $x = a$ as shown in Fig. 8.1.

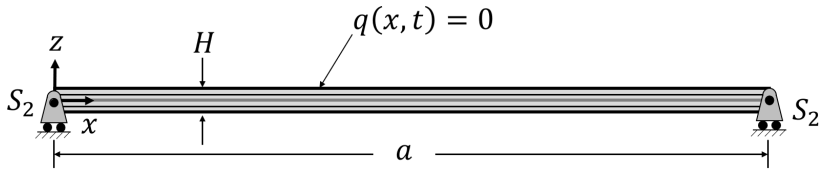


Figure 8.1: Free vibration of a simply supported cross-ply laminate in cylindrical bending

In the case of a cross-ply laminate, the rigidities

$$A_{16} = A_{26} = 0, \qquad B_{16} = B_{26} = 0, \qquad D_{16} = D_{26} = 0 \tag{8.9}$$

The distributed load $q(x,t) = 0$ since we are interested in the natural frequencies and mode shapes for a laminated plate in free vibration.

8.2.1 Displacements, mid-surface strains and curvatures

We assume the following form of the displacements,

$$\begin{aligned} u_0(x,y,t) &= U_m \cos\left(\frac{m\pi x}{a}\right) e^{i\omega t} \\ v_0(x,y,t) &= 0 \\ w_0(x,y,t) &= W_m \sin\left(\frac{m\pi x}{a}\right) e^{i\omega t} \end{aligned} \tag{8.10}$$

where ω is the frequency and m specifies the mode of vibration.

The midsurface strains, obtained using (3.12) and (8.10), are

$$\begin{aligned} \varepsilon_x^0 &= \frac{\partial u_0}{\partial x} = -U_m \left(\frac{m\pi}{a}\right) \sin\left(\frac{m\pi x}{a}\right) e^{i\omega t} \\ \varepsilon_y^0 &= \frac{\partial v_0}{\partial y} = 0 \\ \gamma_{xy}^0 &= \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} = 0 \end{aligned} \tag{8.11}$$

The midsurface curvatures, obtained using (3.13) and (8.10), are

$$
\begin{aligned}
\kappa_x &= -\frac{\partial^2 w_0}{\partial x^2} = W_m \left(\frac{m\pi}{a}\right)^2 \sin\left(\frac{m\pi x}{a}\right) e^{i\omega t} \\
\kappa_y &= -\frac{\partial^2 w_0}{\partial y^2} = 0 \\
\kappa_{xy} &= -2\frac{\partial^2 w_0}{\partial x \partial y} = 0
\end{aligned}
\tag{8.12}
$$

8.2.2 Force and moment resultants

The force resultant N_x and moment resultant M_x are obtained using the laminate rigidities and the mid-surface strains and curvatures,

$$N_x = A_{11}\varepsilon_x^0 + B_{11}\kappa_x = \left(\frac{m\pi}{a}\right)\left[-A_{11}U_m + B_{11}W_m\left(\frac{m\pi}{a}\right)\right]\sin\left(\frac{m\pi x}{a}\right)e^{i\omega t} \tag{8.13a}$$

$$M_x = B_{11}\varepsilon_x^0 + D_{11}\kappa_x = \left(\frac{m\pi}{a}\right)\left[-B_{11}U_m + D_{11}W_m\left(\frac{m\pi}{a}\right)\right]\sin\left(\frac{m\pi x}{a}\right)e^{i\omega t} \tag{8.13b}$$

It is noted that the force resultant $N_{xy} = A_{16}\varepsilon_x^0 + B_{16}\kappa_x = 0$.

8.2.3 Equations of motion

Substituting for N_x from (8.13a) and u_0 from (8.10) into the equation of motion (8.7a)

$$\frac{\partial N_x}{\partial x} = I_0\frac{\partial^2 u_0}{\partial t^2} \tag{8.14}$$

gives

$$\left(\frac{m\pi}{a}\right)^2\left[-A_{11}U_m + B_{11}W_m\left(\frac{m\pi}{a}\right)\right]\cos\left(\frac{m\pi x}{a}\right)e^{i\omega t} = -I_0\omega^2 U_m\cos\left(\frac{m\pi x}{a}\right)e^{i\omega t} \tag{8.15}$$

Since (8.15) must hold for all x and t, we obtain

$$-A_{11}\left(\frac{m\pi}{a}\right)^2 U_m + B_{11}\left(\frac{m\pi}{a}\right)^3 W_m = -I_0\omega^2 U_m \tag{8.16}$$

which can in turn be expressed as

$$\left[A_{11}\left(\frac{m\pi}{a}\right)^2 - I_0\omega^2\right]U_m - B_{11}\left(\frac{m\pi}{a}\right)^3 W_m = 0 \tag{8.17}$$

The equation of motion (8.7b) is identically satisfied since N_{xy} and v_0 are identically zero.

Substitution of M_x from (8.13b) and w_0 from (8.10) into the equation of motion (8.7c)

$$\frac{\partial^2 M_x}{\partial x^2} + \cancelto{0}{q\,(x,t)} = I_0 \frac{\partial^2 w_0}{\partial t^2} - I_2 \frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 w_0}{\partial x^2}\right) \tag{8.18}$$

yields

$$\begin{aligned} -\left(\frac{m\pi}{a}\right)^3 \left[-B_{11} U_m + D_{11} W_m \left(\frac{m\pi}{a}\right)\right] \sin\left(\frac{m\pi x}{a}\right) e^{i\omega t} = \\ -I_0 \omega^2 W_m \sin\left(\frac{m\pi x}{a}\right) e^{i\omega t} - I_2 \omega^2 \left(\frac{m\pi}{a}\right)^2 W_m \sin\left(\frac{m\pi x}{a}\right) e^{i\omega t} \end{aligned} \tag{8.19}$$

Since (8.19) must hold for all x and t, we obtain

$$B_{11}\left(\frac{m\pi}{a}\right)^3 U_m - D_{11}\left(\frac{m\pi}{a}\right)^4 W_m = -I_0\omega^2 W_m - I_2\omega^2\left(\frac{m\pi}{a}\right)^2 W_m \tag{8.20}$$

which can be expressed as

$$-B_{11}\left(\frac{m\pi}{a}\right)^3 U_m + \left\{D_{11}\left(\frac{m\pi}{a}\right)^4 - \left[I_0 + I_2\left(\frac{m\pi}{a}\right)^2\right]\omega^2\right\} W_m = 0 \tag{8.21}$$

8.2.4 Natural frequencies

Equations (8.17) and (8.21) can be written in matrix form as

$$\begin{bmatrix} A_{11}\left(\frac{m\pi}{a}\right)^2 - I_0\omega^2 & -B_{11}\left(\frac{m\pi}{a}\right)^3 \\ -B_{11}\left(\frac{m\pi}{a}\right)^3 & D_{11}\left(\frac{m\pi}{a}\right)^4 - \left[I_0 + I_2\left(\frac{m\pi}{a}\right)^2\right]\omega^2 \end{bmatrix} \cdot \begin{Bmatrix} U_m \\ W_m \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{8.22}$$

For a non-tirivial solution, the determinant of the matrix in (8.22) must vanish. Therefore,

$$\begin{aligned} D_{11}A_{11}\left(\frac{m\pi}{a}\right)^6 - A_{11}\left[I_0 + I_2\left(\frac{m\pi}{a}\right)^2\right]\left(\frac{m\pi}{a}\right)^2\omega^2 - D_{11}I_0\left(\frac{m\pi}{a}\right)^4\omega^2 + \\ I_0\left[I_0 + I_2\left(\frac{m\pi}{a}\right)^2\right]\omega^4 - B_{11}^2\left(\frac{m\pi}{a}\right)^6 = 0 \end{aligned} \tag{8.23}$$

Equation (8.23) can be re-written as

$$\alpha\omega^4 - \beta\omega^2 + \gamma = 0 \tag{8.24}$$

where the coefficients α, β and γ are defined as

$$\begin{aligned} \alpha &= I_0\left[I_0 + I_2\left(\frac{m\pi}{a}\right)^2\right] \\ \beta &= A_{11}\left[I_0 + I_2\left(\frac{m\pi}{a}\right)^2\right]\left(\frac{m\pi}{a}\right)^2 + D_{11}I_0\left(\frac{m\pi}{a}\right)^4 \\ \gamma &= D_{11}A_{11}\left(\frac{m\pi}{a}\right)^6 - B_{11}^2\left(\frac{m\pi}{a}\right)^6 \end{aligned} \tag{8.25}$$

The roots of (8.24) yield the natural frequency ω_m corresponding to mode m

$$\omega_m = \left[\frac{\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}\right]^{\frac{1}{2}} \tag{8.26}$$

Equation (8.26) yields two positive roots, namely $\omega_m^{(1)}$ and $\omega_m^{(2)}$, for each m. Note that the resulting natural frequencies $\omega_m^{(i)}$ have units of rad/sec.

8.2.5 Mode shapes

The mode shape corresponding to one of the roots, say $\omega_m^{(i)}$ ($i = 1$ or 2), is obtained by setting $\omega = \omega_m^{(i)}$ in (8.22) and striking out one of the redundant equations since the determinant is zero,

$$\begin{bmatrix} A_{11}\left(\frac{m\pi}{a}\right)^2 - I_0\left(\omega_m^{(i)}\right)^2 & -B_{11}\left(\frac{m\pi}{a}\right)^3 \\ \cancel{-B_{11}\left(\frac{m\pi}{a}\right)^3} & \cancel{D_{11}\left(\frac{m\pi}{a}\right)^4 - \left[I_0 + I_2\left(\frac{m\pi}{a}\right)^2\right]\left(\omega_m^{(i)}\right)^2} \end{bmatrix} \cdot \begin{Bmatrix} U_m \\ W_m \end{Bmatrix} = \begin{Bmatrix} 0 \\ \cancel{0} \end{Bmatrix} \tag{8.27}$$

It follows from (8.27) that

$$\left[A_{11}\left(\frac{m\pi}{a}\right)^2 - I_0\left(\omega_m^{(i)}\right)^2\right] U_m = B_{11}\left(\frac{m\pi}{a}\right)^3 W_m \tag{8.28}$$

and therefore

$$\frac{U_m}{W_m} = \frac{B_{11}\left(\frac{m\pi}{a}\right)^3}{A_{11}\left(\frac{m\pi}{a}\right)^2 - I_0\left(\omega_m^{(i)}\right)^2} \tag{8.29}$$

Equation (8.29) gives the ratio of the amplitude of u_0 to the amplitude of w_0. The mode shapes for the deflection $w_0(x)$ and in-plane displacement $u_0(x)$ are shown in Fig. 8.2 for $m = 1$.

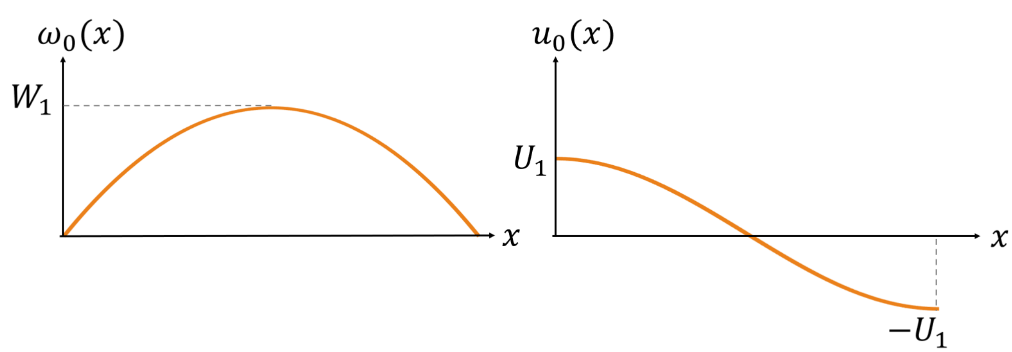


Figure 8.2: Mode shape corresponding to $m = 1$ for a simply supported laminate in cylindrical bending

The mode shapes for $m = 2$ are shown in Fig. 8.3.

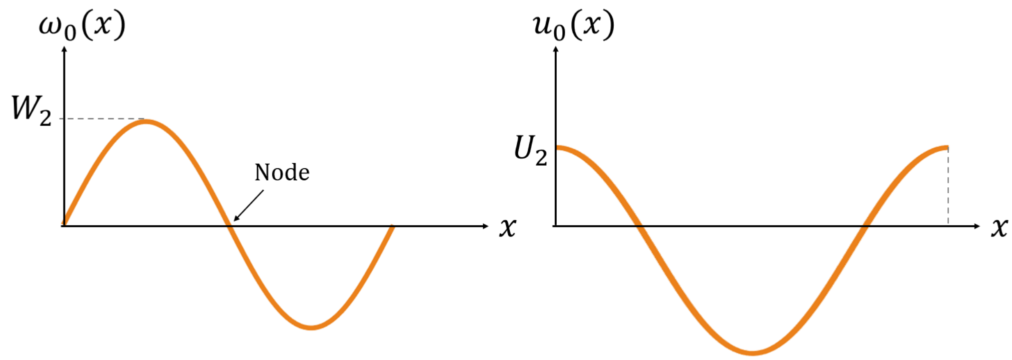


Figure 8.3: Mode shape corresponding to $m = 2$ for a simply supported laminate in cylindrical bending

8.2.6 Symmetric cross-ply laminates

In the case of a symmetric cross-ply laminate, $B_{11} = 0$. Therefore, (8.21) and (8.17) reduce to

$$\left[D_{11}\left(\frac{m\pi}{a}\right)^4 - \left[I_0 + I_2\left(\frac{m\pi}{a}\right)^2\right]\omega^2\right]W_m = 0 \tag{8.30a}$$

$$\left[A_{11}\left(\frac{m\pi}{a}\right)^2 - I_0\omega^2\right]U_m = 0 \tag{8.30b}$$

As is evident from Eqn. (8.30), the amplitude of the out-of-plane deflection W_m is uncoupled from the amplitude of the in-plane displacement U_m.

Equation (8.30a) gives the natural frequency of bending vibration $\omega_m^{(1)}$,

$$\begin{aligned}\omega_m^{(1)} &= \left(\frac{m\pi}{a}\right)^2\sqrt{\frac{D_{11}}{I_0 + I_2\left(\frac{m\pi}{a}\right)^2}} \\ &= \left(\frac{m\pi}{a}\right)^2\sqrt{\frac{D_{11}}{I_0}}\sqrt{\frac{1}{1 + \frac{I_2}{I_0}\left(\frac{m\pi}{a}\right)^2}}\end{aligned} \tag{8.31}$$

Since $I_2 = \rho H^3/12$ and $I_0 = \rho H$, the ratio of the rotary inertia to the areal density is

$$\frac{I_2}{I_0} = \frac{H^2}{12} \tag{8.32}$$

It follows from (8.32) and (8.31) that the natural frequency of bending vibration

$$\omega_m^{(1)} = \left(\frac{m\pi}{a}\right)^2\sqrt{\frac{D_{11}}{I_0}}\sqrt{\frac{1}{1 + \frac{1}{12}\left(\frac{H}{a}\right)^2(m\pi)^2}} \tag{8.33}$$

The normalized fundamental frequency $\omega_m^{(1)}$ is shown in Fig. 8.4 as a function of the length-to-thickness

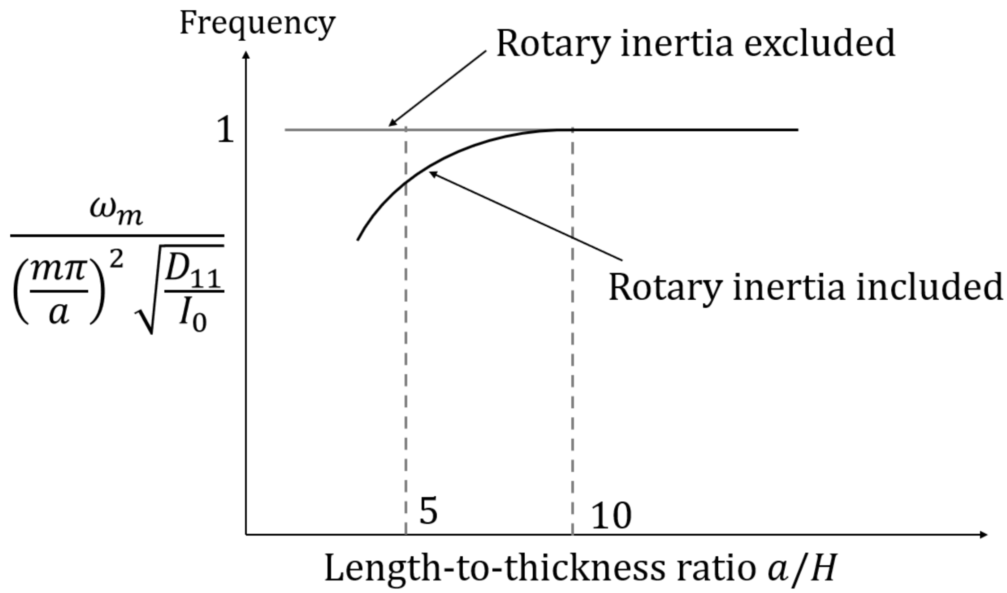


Figure 8.4: Effect of rotary inertia on the fundamental frequency of a laminated plate in cylindrical bending

ratio a/H. As can be seen, the fundamental frequency decreases as the thickness H increases due to the rotary inertia. The effect of rotary inertia on the natural frequency is pronounced for moderately thick or thick plates. In the case of thin plates, the rotary inertia can be neglected and the fundamental frequency can be approximated as

$$\omega_m^{(1)} \approx \left(\frac{m\pi}{a}\right)^2 \sqrt{\frac{D_{11}}{I_0}} \tag{8.34}$$

The natural frequency of in-plane vibration, denoted by $\omega_m^{(2)}$, is obtained from Equation (8.30b)

$$\omega_m^{(2)} = \left(\frac{m\pi}{a}\right) \sqrt{\frac{A_{11}}{I_0}} \tag{8.35}$$

8.3 Free vibration of simply supported cross-ply laminated plates

In this section, we consider the free vibration of a cross-ply laminated rectangular plate that is subjected to S_2 boundary conditions on all four edges as shown in Fig. 8.5. The distributed load $q(x, y, t) = 0$ since we are interested in the natural frequencies of a laminated rectangular plate in free vibration. In the case of a cross-ply laminate,

$$A_{16} = A_{26} = 0, \qquad B_{16} = B_{26} = 0, \qquad D_{16} = D_{26} = 0$$

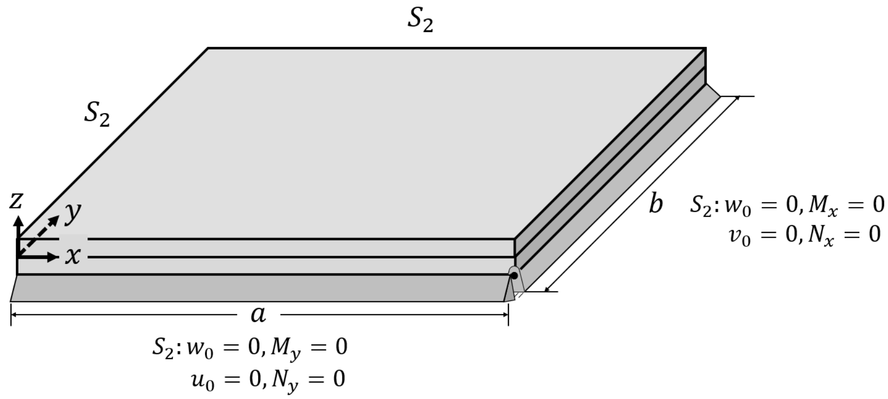


Figure 8.5: Vibration of a simply supported laminated rectangular plate

8.3.1 Displacements, mid-surface strains and curvatures

We assume a Navier solution for the mid-surface displacements of the form

$$\begin{aligned} u_0(x,y,t) &= U_{mn}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{i\omega t} \\ v_0(x,y,t) &= V_{mn}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{i\omega t} \\ w_0(x,y,t) &= W_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{i\omega t} \end{aligned} \tag{8.36}$$

The midsurface strains and curvatures corresponding to the assumed displacements (8.36) are

$$\begin{aligned} \varepsilon_x^0 &= \frac{\partial u_0}{\partial x} = U_{mn}\left(-\frac{m\pi}{a}\right)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}e^{i\omega t} \\ \varepsilon_y^0 &= \frac{\partial v_0}{\partial y} = V_{mn}\left(-\frac{n\pi}{b}\right)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}e^{i\omega t} \\ \gamma_{xy}^0 &= \frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x} = \left[U_{mn}\left(\frac{n\pi}{b}\right)+V_{mn}\left(\frac{m\pi}{a}\right)\right]\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}e^{i\omega t} \\ \kappa_x &= -\frac{\partial^2 w_0}{\partial x^2} = W_{mn}\left(\frac{m\pi}{a}\right)^2\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}e^{i\omega t} \\ \kappa_y &= -\frac{\partial^2 w_0}{\partial y^2} = W_{mn}\left(\frac{n\pi}{b}\right)^2\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}e^{i\omega t} \\ \kappa_{xy} &= -2\frac{\partial^2 w_0}{\partial x\partial y} = -2W_{mn}\left(\frac{mn\pi^2}{ab}\right)\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}e^{i\omega t} \end{aligned} \tag{8.37}$$

8.3.2 Force and moment resultants

The force and moment resultants are obtained by substituting for the mid-surface strains and curvatures from (8.37) into (3.26)

$$\begin{aligned}
N_x &= A_{11}\varepsilon_x^0 + A_{12}\varepsilon_y^0 + B_{11}\kappa_x + B_{12}\kappa_y \\
&= \left[-A_{11}\left(\frac{m\pi}{a}\right)U_{mn} - A_{12}\left(\frac{n\pi}{b}\right)V_{mn} + \left(B_{11}\left(\frac{m\pi}{a}\right)^2 + B_{12}\left(\frac{n\pi}{b}\right)^2\right)W_{mn}\right]\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}e^{i\omega t} && (8.38a)\\
N_y &= A_{12}\varepsilon_x^0 + A_{22}\varepsilon_y^0 + B_{12}\kappa_x + B_{22}\kappa_y \\
&= \left[-A_{12}\left(\frac{m\pi}{a}\right)U_{mn} - A_{22}\left(\frac{n\pi}{b}\right)V_{mn} + \left(B_{12}\left(\frac{m\pi}{a}\right)^2 + B_{22}\left(\frac{n\pi}{b}\right)^2\right)W_{mn}\right]\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}e^{i\omega t} && (8.38b)\\
N_{xy} &= A_{66}\gamma_{xy}^0 + B_{66}\kappa_{xy} \\
&= \left[\left(U_{mn}\left(\frac{n\pi}{b}\right) + V_{mn}\left(\frac{m\pi}{a}\right)\right)A_{66} - 2B_{66}\left(\frac{mn\pi^2}{ab}\right)W_{mn}\right]\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}e^{i\omega t} && (8.38c)\\
M_x &= B_{11}\varepsilon_x^0 + B_{12}\varepsilon_y^0 + D_{11}\kappa_x + D_{12}\kappa_y \\
&= \left[-B_{11}\left(\frac{m\pi}{a}\right)U_{mn} - B_{12}\left(\frac{n\pi}{b}\right)V_{mn} + \left(D_{11}\left(\frac{m\pi}{a}\right)^2 + D_{12}\left(\frac{n\pi}{b}\right)^2\right)W_{mn}\right]\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}e^{i\omega t} && (8.38d)\\
M_y &= B_{12}\varepsilon_x^0 + B_{22}\varepsilon_y^0 + D_{12}\kappa_x + D_{22}\kappa_y \\
&= \left[-B_{12}\left(\frac{m\pi}{a}\right)U_{mn} - B_{22}\left(\frac{n\pi}{b}\right)V_{mn} + \left(D_{12}\left(\frac{m\pi}{a}\right)^2 + D_{22}\left(\frac{n\pi}{b}\right)^2\right)W_{mn}\right]\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}e^{i\omega t} && (8.38e)\\
M_{xy} &= B_{66}\gamma_{xy}^0 + D_{66}\kappa_{xy} \\
&= \left[\left(U_{mn}\left(\frac{n\pi}{b}\right) + V_{mn}\left(\frac{m\pi}{a}\right)\right)B_{66} - 2D_{66}\left(\frac{mn\pi^2}{ab}\right)W_{mn}\right]\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}e^{i\omega t} && (8.38f)
\end{aligned}$$

8.3.3 Equations of motion

The equations of motions for a laminated plate are,

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = I_0\frac{\partial^2 u_0}{\partial t^2} \tag{8.39a}$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = I_0\frac{\partial^2 v_0}{\partial t^2} \tag{8.39b}$$

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x\partial y} + \frac{\partial^2 M_y}{\partial y^2} + \cancelto{0}{q(x,y,t)} = I_0\frac{\partial^2 w_0}{\partial t^2} - I_2\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial^2 w_0}{\partial y^2}\right) \tag{8.39c}$$

The time derivatives of the mid-surface displacements and curvatures on the right hand side of equations of motion (8.36) are

$$
\begin{aligned}
I_0 \frac{\partial^2 u_0}{\partial t^2} &= -I_0 \omega^2 U_{mn} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{i\omega t} \\
I_0 \frac{\partial^2 v_0}{\partial t^2} &= -I_0 \omega^2 V_{mn} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{i\omega t} \\
I_0 \frac{\partial^2 w_0}{\partial t^2} &= -I_0 \omega^2 W_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{i\omega t} \\
I_2 \frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 w_0}{\partial x^2} + \frac{\partial^2 w_0}{\partial y^2}\right) &= I_2 \omega^2 W_{mn} \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right] \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{i\omega t}
\end{aligned} \tag{8.40}
$$

Substituting the force and moment resultants (8.38), and time derivatives of the mid-surface displacements and curvatures (8.40), into the equations of motion (8.39) and requiring that the equations hold for all x, y and t, yields the following system of equations

$$
\begin{aligned}
K_{11} U_{mn} + K_{12} V_{mn} + K_{13} W_{mn} &= M_{11} \omega^2 U_{mn} \\
K_{12} U_{mn} + K_{22} V_{mn} + K_{23} W_{mn} &= M_{22} \omega^2 V_{mn} \\
K_{13} U_{mn} + K_{23} V_{mn} + K_{33} W_{mn} &= M_{33} \omega^2 W_{mn}
\end{aligned} \tag{8.41}
$$

where the stiffness constants K_{ij} were previously defined in equations (6.41), (6.45) and (6.49) and the mass constants M_{11}, M_{22} and M_{33} are defined as follows

$$
M_{11} = I_0, \qquad M_{22} = I_0, \qquad M_{33} = I_0 + I_2 \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right] \tag{8.42}
$$

Equations (8.41) can be written in matrix form as

$$
\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{12} & K_{22} & K_{23} \\ K_{13} & K_{23} & K_{33} \end{bmatrix} \cdot \begin{Bmatrix} U_{mn} \\ V_{mn} \\ W_{mn} \end{Bmatrix} = \omega^2 \begin{bmatrix} M_{11} & 0 & 0 \\ 0 & M_{22} & 0 \\ 0 & 0 & M_{33} \end{bmatrix} \cdot \begin{Bmatrix} U_{mn} \\ V_{mn} \\ W_{mn} \end{Bmatrix} \tag{8.43}
$$

8.3.4 Natural frequencies

Equation (8.43) is an eigenvalue problem for the natural frequencies ω for each m and n. It can be written in compact form as follows

$$
[K] \cdot \{d\} = \lambda [M] \{d\} \tag{8.44}
$$

where,

$$[K] = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{12} & K_{22} & K_{23} \\ K_{13} & K_{23} & K_{33} \end{bmatrix}, \quad [M] = \begin{bmatrix} M_{11} & 0 & 0 \\ 0 & M_{22} & 0 \\ 0 & 0 & M_{33} \end{bmatrix}, \quad \lambda = \omega^2, \quad \{d\} = \begin{Bmatrix} U_{mn} \\ V_{mn} \\ W_{mn} \end{Bmatrix} \tag{8.45}$$

Equation (8.44) is a generalized eigenvalue problem which can be solved numerically to obtain the eigenvalues λ and hence the natural frequency $\omega = \sqrt{\lambda}$ for each combination of m and n. In general, the eigenvalue problem will yield three eigenvalues, namely $\lambda_{mn}^{(1)}$, $\lambda_{mn}^{(2)}$ and $\lambda_{mn}^{(3)}$. The corresponding natural frequencies are

$$\omega_{mn}^{(i)} = \sqrt{\lambda_{mn}^{(i)}} \tag{8.46}$$

The bending natural frequency ω_{mn} is typically the smallest of the three natural frequencies $\omega_{mn}^{(i)}$, i.e.,

$$\omega_{mn} = \min \left\{ \omega_{mn}^{(1)}, \omega_{mn}^{(2)}, \omega_{mn}^{(3)} \right\} \tag{8.47}$$

MATLAB code snippet

Use the following commands to calculate the eigenvalues and eigenvectors of the generalized eigenvalue problem (8.44) numerically.

```
» [d,lambda] = eig(K,M)
» omega = sqrt(lambda)
» omega_mn = min(diag(omega))
```

This will yield a 3×3 diagonal matrix *lambda* of eigenvalues and a 3×3 matrix *d* of eigenvectors. The diagonal values of the 3×3 matrix *omega* give three natural frequencies $\omega_{mn}^{(1)}$, $\omega_{mn}^{(2)}$ and $\omega_{mn}^{(3)}$. The i^{th} column of the matrix d are the amplitudes $[U_{mn}^{(i)}, V_{mn}^{(i)}, W_{mn}^{(i)}]^T$.

We can tabulate the bending natural frequency ω_{mn} values for the various mode shapes corresponding to m, n to find the fundamental frequency,

m \ n	1	2	3	...
1	ω_{11}	ω_{12}	ω_{13}	...
2	ω_{21}	ω_{22}	ω_{23}	...
3	ω_{31}	ω_{32}	ω_{33}	...
⋮				

Table 8.1: Tabulated ω_{mn} to find the fundamental frequency

The fundamental frequency ω_0

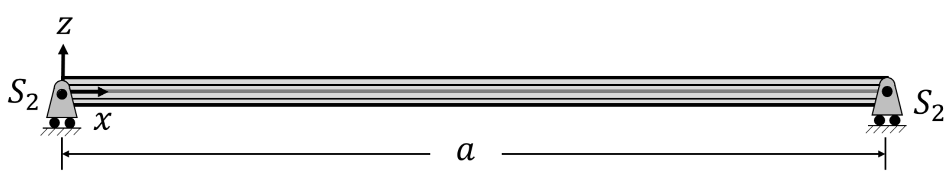
$$\omega_0 = \min_{m,n} \omega_{mn} \tag{8.48}$$

In general, ω_{11} need not be the fundamental frequency since the smallest natural frequency might occur for values other than $m = n = 1$ depending on the laminate dimensions and stacking sequence.

Exercises

Use the representative properties in Sec. 1.10.1 for unidirectional carbon/epoxy composites and the representative properties in Sec. 1.10.2 for fabric-reinforced carbon/epoxy composites. Assume that the ply thickness h of the unidirectional and fabric-reinforced laminae are 0.2 mm and 0.4 mm, respectively. Use the Tsai-Wu theory for failure analysis.

8.1 Consider the free vibration in cylindrical bending of a simply supported $[0_2/90_2]$ laminate made of unidirectional carbon fiber-reinforced plies. The laminate is of width $a = 0.5$ m and the edges $x = 0$ and $x = a$ are simply supported with S_2 boundary conditions.



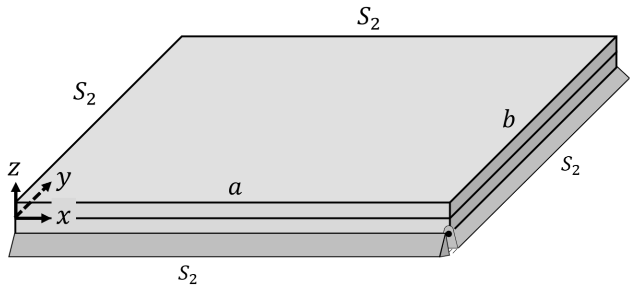
Assume a solution for the mid-surface displacements of the form in (8.10).

(a) Determine the natural frequencies $\omega_m^{(1)}$ and $\omega_m^{(2)}$ and the ratio U_m/W_m for the mode of vibration corresponding to $m = 1$. Plot the corresponding normalized mode shapes $u_0(x)/w_0(a/2)$ and $w_0(x)/w_0(a/2)$ as a function of x for each natural frequency. Discuss the significance of the two modes of vibration and what they represent.

(b) Determine the natural frequencies $\omega_m^{(1)}$ and $\omega_m^{(2)}$ and the ratio U_m/W_m for the mode of vibration corresponding to $m = 2$. Plot the corresponding normalized mode shapes $u_0(x)/w_0(a/4)$ and $w_0(x)/w_0(a/4)$ as a function of x for each natural frequency. Do the magnitudes of the natural frequencies for $m = 2$ make sense relative to the natural frequencies for $m = 1$?

8.2 Consider the free vibration of a $[0_4/90_4]$ cross-ply laminated rectangular plate made of unidirectional carbon fiber-reinforced plies. The length and width of the laminated plate are $a = 0.3$ m and $b = 0.5$ m. All four edges of the plate are simply supported with S_2 boundary conditions.

Using a Navier solution for the mid-surface displacements of the form (8.36),



(a) Calculate the three natural frequencies $\omega_{mn}^{(i)}$ and the relative amplitudes $U_{mn}^{(i)}/W_{mn}^{(i)}$ and $V_{mn}^{(i)}/W_{mn}^{(i)}$ corresponding to $m = 1$ and $n = 1$. Discuss the significance of the three modes of vibration and what they represent. Determine the smallest of the three natural frequencies, i.e., $\omega_{11} = \min_i \omega_{11}^{(i)}$, and discuss the mode of vibration that it corresponds to.

(b) Evaluate the natural frequencies ω_{12}, ω_{21} and ω_{22} and determine the fundamental frequency ω_0 of the laminated rectangular plate. What mode of vibration does ω_0 correspond to?

Part III: First-order Shear Deformation Theory and its Applications

First Order Shear Deformation Theory 9

When analyzing moderately thick to thick laminated plates and sandwich composites, we need to take into account the affects of transverse shear deformation. In this chapter, we will derive the governing equations for the first-order shear deformation theory.

9.1 Kinematics

The first order shear deformation theory is based on the assumption that plane sections, originally perpendicular to the midsurface, remain plane but they need not remain perpendicular to the midsurface after deformation as shown in Fig. 9.1.

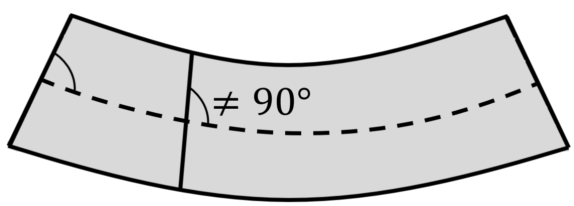


Figure 9.1: Rotation of normal due to shear deformation

The assumed displacement field for the first order shear deformation theory is of the form,

$$\begin{aligned} u(x,y,z,t) &= u_0(x,y,t) + z\phi_x(x,y,t) \\ v(x,y,z,t) &= v_0(x,y,t) + z\phi_y(x,y,t) \\ w(x,y,z,t) &= w_0(x,y,t) \end{aligned} \tag{9.1}$$

where u_0, v_0 and w_0 are the mid-surface displacements. When we take the partial derivatives of the in-plane displacements u and v with respect to z we obtain

$$\frac{\partial u}{\partial z} = \phi_x, \quad \frac{\partial v}{\partial z} = \phi_y \tag{9.2}$$

which indicates that ϕ_x and ϕ_y are the rotations of a transverse normal in the $x-z$ and $y-z$ planes, respectively, as shown in Fig. 9.2. Note that in general, that the rotation of a normal need not equal the slope of the mid-surface, i.e.,

$$\phi_x \neq -\frac{\partial w_0}{\partial x}, \quad \phi_y \neq -\frac{\partial w_0}{\partial y} \tag{9.3}$$

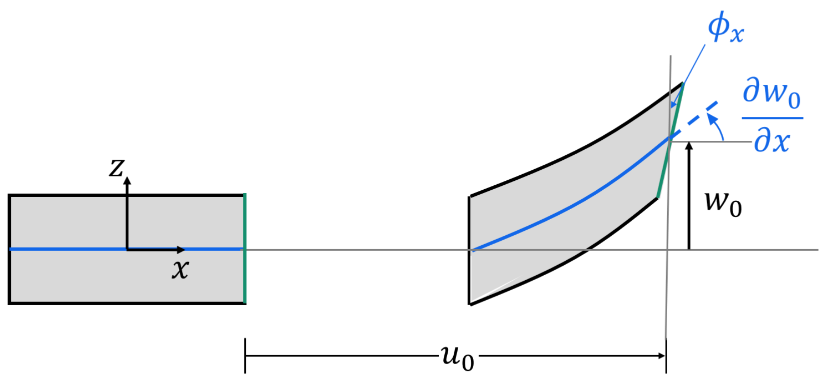


Figure 9.2: Kinematics of the first order shear deformation theory

In other words, the normal need to be perpendicular to the mid-surface after deformation. The reduction in angle between the normal and the midsurface in the $x-z$ plane is the transverse shear strain γ_{xz}, i.e.

$$\gamma_{xz} = \phi_x + \frac{\partial w_0}{\partial x} \tag{9.4}$$

In general, the transverse shear strains γ_{xz} and γ_{yz} need not be zero in the first order shear deformation theory.

9.1.1 Strains

Given the assumed displacement field in equation (9.1), the strains can be calculated using equation (1.10) as,

$$\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x} = \frac{\partial u_0}{\partial x} + z\frac{\partial \phi_x}{\partial x} \\ \varepsilon_y &= \frac{\partial v}{\partial y} = \frac{\partial v_0}{\partial y} + z\frac{\partial \phi_y}{\partial y} \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + z\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right) \end{aligned} \tag{9.5}$$

The strains can be expressed as

$$\begin{aligned} \varepsilon_x &= \varepsilon_x^0 + z\kappa_x \\ \varepsilon_y &= \varepsilon_y^0 + z\kappa_y \\ \gamma_{xy} &= \gamma_{xy}^0 + z\kappa_{xy} \end{aligned} \tag{9.6}$$

where ε_x^0, ε_y^0 and γ_{xy}^0 are the mid-surface strains and have the same definitions as before, i.e.,

$$\varepsilon_x^o = \frac{\partial u_o}{\partial x}. \quad \varepsilon_y^o = \frac{\partial v_o}{\partial y}, \quad \gamma_{xy}^o = \frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x} \tag{9.7}$$

and the κ's are related to the rotations of the normal to the midsurface as follows,

$$\kappa_x = \frac{\partial \phi_x}{\partial x}, \quad \kappa_y = \frac{\partial \phi_y}{\partial y}, \quad \kappa_{xy} = \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \tag{9.8}$$

In the case of the first order shear deformation theory, the transverse shear strains are in general non-zero and are defined as follows,

$$\begin{aligned} \gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = \phi_y + \frac{\partial w_0}{\partial y} \\ \gamma_{xz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \phi_x + \frac{\partial w_0}{\partial x} \end{aligned} \tag{9.9}$$

9.2 Force and moment resultants

9.2.1 In-plane force and moment resultants

The three-dimensional stress-strain for an off-axis lamina in the $x - y - z$ coordinate system are

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \overline{S}_{11} & \overline{S}_{12} & \overline{S}_{13} & 0 & 0 & \overline{S}_{16} \\ \overline{S}_{12} & \overline{S}_{22} & \overline{S}_{23} & 0 & 0 & \overline{S}_{26} \\ \overline{S}_{13} & \overline{S}_{23} & \overline{S}_{33} & 0 & 0 & \overline{S}_{36} \\ 0 & 0 & 0 & \overline{S}_{44} & \overline{S}_{45} & 0 \\ 0 & 0 & 0 & \overline{S}_{45} & \overline{S}_{55} & 0 \\ \overline{S}_{16} & \overline{S}_{26} & \overline{S}_{36} & 0 & 0 & \overline{S}_{66} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} \tag{9.10}$$

In the case of the first-order shear deformation theory, it is assumed that the transverse normal stress σ_z is zero. Setting $\sigma_z = 0$ in Eqn. (9.10) gives the following reduced constitutive relationship for the in-plane stresses and strains

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \left[\overline{S}\right] \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \tag{9.11}$$

where $\left[\overline{S}\right]$ is identical to the plane stress-reduced compliance matrix for a lamina in the classical laminated plate theory. The inverse relationships are

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \left[\overline{Q}\right] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{9.12}$$

where $\left[\overline{Q}\right]$ is the off-axis reduced stiffness matrix for a lamina. The in-plane force resultants and the moment resultants follow as,

$$\{N\} = \int_{-H/2}^{H/2} \{\sigma\}\, dz, \quad \{M\} = \int_{-H/2}^{H/2} \{\sigma\}\, z dz \tag{9.13}$$

The resulting laminate constitutive relations are the same as before,

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \cdot \begin{Bmatrix} \varepsilon^0 \\ \kappa \end{Bmatrix} \tag{9.14}$$

where the laminate rigidities $[A]$, $[B]$ and $[D]$ are the same as those for the classical laminated plate theory.

9.2.2 Transverse shear force resultants

In the case of an off-axis layer, the structure of the elastic stiffness tensor resembles that of a monoclinic material. The 6×6 elastic stiffness matrix for an off-axis layer in the $x-y-z$ coordinate system is

$$\left[\overline{C}\right] = \begin{bmatrix} \overline{C}_{11} & \overline{C}_{12} & \overline{C}_{13} & 0 & 0 & \overline{C}_{16} \\ \overline{C}_{12} & \overline{C}_{22} & \overline{C}_{23} & 0 & 0 & \overline{C}_{26} \\ \overline{C}_{13} & \overline{C}_{23} & \overline{C}_{33} & 0 & 0 & \overline{C}_{36} \\ 0 & 0 & 0 & \overline{C}_{44} & \overline{C}_{45} & 0 \\ 0 & 0 & 0 & \overline{C}_{45} & \overline{C}_{55} & 0 \\ \overline{C}_{16} & \overline{C}_{26} & \overline{C}_{36} & 0 & 0 & \overline{C}_{66} \end{bmatrix} \tag{9.15}$$

where,

$$\begin{aligned} \overline{C}_{44} &= m^2 C_{44} + n^2 C_{55}, \\ \overline{C}_{55} &= m^2 C_{55} + n^2 C_{44}, \\ \overline{C}_{45} &= (C_{55} - C_{44})\, mn \end{aligned} \tag{9.16}$$

and $m = \cos\theta$, $n = \sin\theta$. Here C_{44} and C_{55} are the transverse shear moduli in the prinicipal material coordinate system, i.e.,

$$C_{44} = G_{23}, \quad C_{55} = G_{13} \tag{9.17}$$

Therefore, the off-axis elastic moduli $\overline{C}_{44}$, $\overline{C}_{55}$ and $\overline{C}_{45}$ can be expressed in terms of the transverse shear moduli as follows

$$\begin{aligned}\overline{C}_{44} &= m^2 G_{23} + n^2 G_{13},\\ \overline{C}_{55} &= m^2 G_{13} + n^2 G_{23},\\ \overline{C}_{45} &= (G_{13} - G_{23})\, mn\end{aligned} \tag{9.18}$$

It follows from equation (9.15) that,

$$\begin{Bmatrix}\tau_{yz}\\ \tau_{xz}\end{Bmatrix} = \begin{bmatrix}\overline{C}_{44} & \overline{C}_{45}\\ \overline{C}_{45} & \overline{C}_{55}\end{bmatrix} \cdot \begin{Bmatrix}\gamma_{yz}\\ \gamma_{xz}\end{Bmatrix} \tag{9.19}$$

Since the shear strains are constant through the plate thickness, the transverse shear stresses are constant through the thickness of each ply.

The transverse shear force resultants are obtained by integrating the transverse shear stresses through the thickness of the laminate

$$\begin{Bmatrix}V_y\\ V_x\end{Bmatrix} = \int_{-H/2}^{H/2} \begin{Bmatrix}\tau_{yz}\\ \tau_{xz}\end{Bmatrix} dz \tag{9.20}$$

It follows from equations (9.20) and (9.19) that,

$$\begin{Bmatrix}V_y\\ V_x\end{Bmatrix} = \int_{-H/2}^{H/2} \begin{bmatrix}\overline{C}_{44} & \overline{C}_{45}\\ \overline{C}_{45} & \overline{C}_{55}\end{bmatrix} \cdot \begin{Bmatrix}\gamma_{yz}\\ \gamma_{xz}\end{Bmatrix} dz \tag{9.21}$$

which can be written as

$$\begin{Bmatrix}V_y\\ V_x\end{Bmatrix} = K \begin{bmatrix}A_{44} & A_{45}\\ A_{45} & A_{55}\end{bmatrix} \cdot \begin{Bmatrix}\gamma_{yz}\\ \gamma_{xz}\end{Bmatrix} \tag{9.22}$$

where A_{44}, A_{55} and A_{45} are the transverse shear rigidities that are define as follows

$$A_{ij} = \int_{-H/2}^{H/2} \overline{C}_{ij} dz = \sum_{k=1}^{N} (z_{k+1} - z_k)\, \overline{C}_{ij}^{(k)}, \quad i, j = 4, 5 \tag{9.23}$$

and K is a shear correction factor that is introduced to account for the discrepancy between the actual stress state and the constant stress state assumed by the first order shear deformation theory. Typically, $K \leq 1$ which amounts to reducing the plate transverse shear stiffnesses. It is noted that using $K = 1$ will underestimate the deflection.

9.2.3 Shear correction factor

The shear correction factor K is computed such that the strain energy due to transverse shear stresses equals the strain energy due to the true transverse shear stresses predicted by the elasticity theory.

Consider a homogeneous beam with rectangular cross section of width b and height H. The actual shear stress distribution through the height of the beam from strength of materials is,

$$\tau_{xz}^{(a)} = \frac{6V}{bH}\left[\left(\frac{1}{2}\right)^2 - \left(\frac{z}{H}\right)^2\right], \quad -\frac{H}{2} \le z \le \frac{H}{2} \tag{9.24}$$

where V is the transverse shear force.

The strain energy per unit length of the beam is,

$$\tilde{U} = \frac{1}{2}\int_A \tau_{xz}\gamma_{xz}\,dA = \frac{b}{2}\int_{-H/2}^{H/2} \tau_{xz}\gamma_{xz}\,dz \tag{9.25}$$

The strain energy due to the actual stress distribution follows from equations (9.24) and (9.25) as,

$$\begin{aligned}
\tilde{U}^{(a)} &= \frac{b}{2}\int_{-H/2}^{H/2} \tau_{xz}\frac{\tau_{xz}}{G}\,dz \\
&= \frac{b}{2G}\int_{-H/2}^{H/2} \frac{36V^2}{b^2H^2}\left[\left(\frac{1}{2}\right)^2 - \left(\frac{z}{H}\right)^2\right]dz \\
&= \frac{18V^2}{GbH^2}\int_{-H/2}^{H/2}\left[\frac{1}{16} - \frac{1}{2}\frac{z^2}{H^2} + \frac{z^4}{H^4}\right]dz \\
&= \frac{18V^2}{GbH^2}\left[\frac{z}{16} - \frac{z^3}{6H^2} + \frac{z^5}{5H^4}\right]_{-H/2}^{H/2} \\
&= \frac{18V^2}{GbH^2}\cdot 2\cdot\left[\frac{H}{32} - \frac{H}{48} + \frac{H}{160}\right] \\
&= \frac{18V^2}{GbH^2}\cdot 2\cdot\frac{H}{60} = \frac{3V^2}{5GbH}
\end{aligned} \tag{9.26}$$

In the case of an isotropic beam, the strain energy due to constant shear stress through the thickness is

$$A_{55} = \int_{-H/2}^{H/2} \overline{C}_{55}\,dz = \int_{-H/2}^{H/2} G\,dz = GH \tag{9.27}$$

In the case of the first-order shear deformation theory, the shear force resultant

$$V_x = KA_{55}\gamma_{xz}^{(f)} \tag{9.28}$$

Since the shear force resultant is the shear force per unit width,

$$\frac{V}{b} = K\,(GH)\,\gamma_{xz}^{(f)} \quad \Rightarrow \quad \gamma_{xz}^{(f)} = \frac{V}{KGbH} \tag{9.29}$$

The shear force resultant can also be obtained by integrating the shear stress through the thickness, i.e.,

$$V_x = \int_{-H/2}^{H/2} \tau_{xz}^{(f)}\,dz \tag{9.30}$$

Since the transverse shear stress is assumed to be constant through the thickness,

$$\frac{V}{b} = \tau_{xz}^{(f)} H \quad \Rightarrow \quad \tau_{xz}^{(f)} = \frac{V}{bH} \tag{9.31}$$

Therefore, the strain energy per unit length for a constant stress distribution is,

$$\begin{aligned}\tilde{U}^{(f)} &= \frac{b}{2}\int_{-H/2}^{H/2} \tau_{xz}^{(f)}\gamma_{xz}^{(f)}\,dz \\ &= \frac{b}{2}\int_{-H/2}^{H/2} \frac{V}{bH}\,\frac{V}{KGbH}\,dz \\ &= \frac{b}{2}\cdot\frac{V^2}{KGb^2H^2}\cdot H = \frac{V^2}{2KGbH}\end{aligned} \tag{9.32}$$

Equating the strain energy densities from equations (9.32) and (9.26),

$$\tilde{U}^{(a)} = \tilde{U}^{(f)} \quad \Rightarrow \quad \frac{3V^2}{5GbH} = \frac{V^2}{2KGbH} \tag{9.33}$$

from which it follows that

$$K = \frac{5}{6} \tag{9.34}$$

Thus, by equating the strain energy densities of the assumed constant transverse shear stress of the first-order shear deformation theory and the actual parabolic variation of transverse shear stress for an isotropic beam, we obtain a shear correction factor of $K = 5/6$

9.3 Equations of motion and boundary conditions

As discussed earlier in Chapter 4, the three-dimensional equations of motion are integrated through the thickness to obtain the equations of motion for a laminated plate in terms of the force and moment resultants. Following the procedure in Sec.4.2.2 we obtain the following five equations of motion,

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = \int_{-H/2}^{H/2} \rho \frac{\partial^2 u}{\partial t^2} dz \tag{9.35a}$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = \int_{-H/2}^{H/2} \rho \frac{\partial^2 v}{\partial t^2} dz \tag{9.35b}$$

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + q\,(x, y, t) = \int_{-H/2}^{H/2} \rho \frac{\partial^2 w}{\partial t^2} dz \tag{9.35c}$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - V_x = \int_{-H/2}^{H/2} \rho \frac{\partial^2 u}{\partial t^2} z dz \tag{9.35d}$$

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - V_y = \int_{-H/2}^{H/2} \rho \frac{\partial^2 v}{\partial t^2} z dz \tag{9.35e}$$

The time derivatives terms on the right hand side of equations (9.35) can be expressed in terms of the mid-surface displacements and rotations of the normal as follows,

$$\begin{aligned}\int_{-H/2}^{H/2} \rho \frac{\partial^2 u}{\partial t^2} dz &= \int_{-H/2}^{H/2} \rho \left(\frac{\partial^2 u_0}{\partial t^2} + z \frac{\partial^2 \phi_x}{\partial t^2} \right) dz \\ &= I_0 \frac{\partial^2 u_0}{\partial t^2} + I_1 \frac{\partial^2 \phi_x}{\partial t^2}\end{aligned} \tag{9.36a}$$

$$\int_{-H/2}^{H/2} \rho \frac{\partial^2 v}{\partial t^2} dz = I_0 \frac{\partial^2 v_0}{\partial t^2} + I_1 \frac{\partial^2 \phi_y}{\partial t^2} \tag{9.36b}$$

$$\int_{-H/2}^{H/2} \rho \frac{\partial^2 w}{\partial t^2} dz = I_0 \frac{\partial^2 w_0}{\partial t^2} \tag{9.36c}$$

$$\int_{-H/2}^{H/2} \rho \frac{\partial^2 u}{\partial t^2} z dz = I_1 \frac{\partial^2 u_0}{\partial t^2} + I_2 \frac{\partial^2 \phi_x}{\partial t^2} \tag{9.36d}$$

$$\int_{-H/2}^{H/2} \rho \frac{\partial^2 v}{\partial t^2} z dz = I_1 \frac{\partial^2 v_0}{\partial t^2} + I_2 \frac{\partial^2 \phi_y}{\partial t^2} \tag{9.36e}$$

The equations of motion for the first order shear deformation theory follow from equations (9.35) and (9.36),

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = I_0 \frac{\partial^2 u_0}{\partial t^2} + I_1 \frac{\partial^2 \phi_x}{\partial t^2} \tag{9.37a}$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = I_0 \frac{\partial^2 v_0}{\partial t^2} + I_1 \frac{\partial^2 \phi_y}{\partial t^2} \tag{9.37b}$$

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + q\,(x, y, t) = I_0 \frac{\partial^2 w_0}{\partial t^2} \tag{9.37c}$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - V_x = I_1 \frac{\partial^2 u_0}{\partial t^2} + I_2 \frac{\partial^2 \phi_x}{\partial t^2} \tag{9.37d}$$

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - V_y = I_1 \frac{\partial^2 v_0}{\partial t^2} + I_2 \frac{\partial^2 \phi_y}{\partial t^2} \tag{9.37e}$$

The five partial differential equations need to be solved to obtain u_0, v_0, w_0, ϕ_x and ϕ_y. As in the case of the classical laminated plate theory, if all the layers have the same density ρ,

$$I_0 = \rho H, \quad I_1 = 0, \quad I_2 = \frac{\rho H^3}{12} \tag{9.38}$$

where H is the thickness of the laminated plate.

9.3.1 Boundary conditions

At $x = 0, a$, the following boundary conditions need to be specified,

$$\begin{array}{ccc} u_0 & \text{or} & N_x \\ v_0 & \text{or} & N_{xy} \\ w_0 & \text{or} & V_x \\ \phi_x & \text{or} & M_x \\ \phi_y & \text{or} & M_{xy} \end{array} \tag{9.39}$$

9.3.2 Clamped edges

If the laminate is clamped at $x = 0$ or $x = a$, the mid-surface displacements are zero. In addition, the normal is restrained against rotation at the clamped edges. Therefore,

$$u_0 = v_0 = w_0 = 0, \ \phi_x = \phi_y = 0 \tag{9.40}$$

Note that $\partial w_0/\partial x$ need not be 0 at a clamped edge.

9.3.3 Free edges

If the laminate is free at $x = 0$ or $x = a$, then the force and moment resultants are zero. Therefore,

$$N_x = N_{xy} = 0, \; V_x = 0, \; M_x = M_{xy} = 0 \tag{9.41}$$

9.3.4 Simply supported edges

There are several different types of simply supported boundary conditions that can be applied at the edges $x = 0$ and $x = a$. An all cases, the deflection w_0 and moment M_x are zero. For example,

$$v_0 = w_0 = 0, \; M_x = 0, \; N_x = 0, \; \phi_y = 0 \tag{9.42}$$

9.4 Cylindrical bending of symmetric cross-ply laminates

Let's consider the cylindrical bending of a symmetric cross-ply laminate as shown in Fig. 9.3.

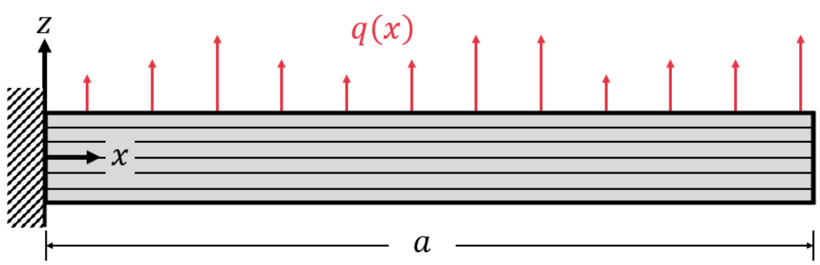


Figure 9.3: Cylindrical bending of a symmetric cross-ply laminate

In the case of a symmetric cross-ply laminate, the following rigidities are identically zero

$$A_{16} = A_{26} = 0, \quad A_{45} = 0, \quad B_{ij} = 0, \quad D_{16} = D_{26} = 0 \tag{9.43}$$

In the case of cylindrical bending, we assume that

$$\frac{\partial\left(\cdot\right)}{\partial y} = 0, \quad v = 0 \tag{9.44}$$

9.4.1 Displacements

Since the laminate is symmetric and cross-ply, it is assumed that only the transverse deflection w_0 and rotation ϕ_x of the normal in the $x-z$ plane are non-zero, i.e.,

$$u_0(x) = v_0(x) = 0, \quad w_0 = w_0(x), \quad \phi_x = \phi_x(x), \quad \phi_y(x) = 0 \tag{9.45}$$

9.4.2 Strains

The strains follow from the definition (9.7) and the assumed forms (9.45) for the displacements and rotations as,

$$\varepsilon_x^0 = \frac{\partial u_0}{\partial x} = 0 \tag{9.46a}$$

$$\varepsilon_y^0 = \frac{\partial v_0}{\partial y} = 0 \tag{9.46b}$$

$$\gamma_{xy}^0 = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} = 0 \tag{9.46c}$$

$$\gamma_{yz} = \phi_y + \frac{\partial w_0}{\partial y} = 0 \tag{9.46d}$$

$$\gamma_{xz} = \phi_x + \frac{\partial w_0}{\partial x} \tag{9.46e}$$

The κ's follow from (9.8) and (9.45),

$$\kappa_x = \frac{\partial \phi_x}{\partial x} \tag{9.47a}$$

$$\kappa_y = \frac{\partial \phi_y}{\partial y} = 0 \tag{9.47b}$$

$$\kappa_{xy} = \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} = 0 \tag{9.47c}$$

9.4.3 Force and moment resultants

The in-plane force resultants and moment resultants follow from (9.14), (9.46) and (9.47)

$$N_x = N_y = N_{xy} = 0 \tag{9.48a}$$

$$M_x = D_{11}\kappa_x = D_{11}\frac{\partial \phi_x}{\partial x} \tag{9.48b}$$

$$M_y = D_{12}\kappa_x = D_{12}\frac{\partial \phi_x}{\partial x} \tag{9.48c}$$

$$M_{xy} = 0 \tag{9.48d}$$

The transverse shear force resultants are obtained using (9.22) and (9.46)

$$\begin{Bmatrix} V_y \\ V_x \end{Bmatrix} = K \begin{bmatrix} A_{44} & \cancelto{0}{A_{45}} \\ \cancelto{0}{A_{45}} & A_{55} \end{bmatrix} \cdot \begin{Bmatrix} \cancelto{0}{\gamma_{yz}} \\ \gamma_{xz} \end{Bmatrix} \tag{9.49}$$

from which it follows that

$$V_y = KA_{44}\cancelto{0}{\gamma_{yz}} = 0 \tag{9.50a}$$

$$V_x = KA_{55}\gamma_{xz} = KA_{55}\left(\phi_x + \frac{\partial w_0}{\partial x}\right) \tag{9.50b}$$

9.4.4 Equilibrium equations

The equilibrium equations (9.37) simplify to,

$$\frac{\partial \cancelto{0}{N_x}}{\partial x} + \frac{\partial \cancelto{0}{N_{xy}}}{\partial y} = 0 \quad \checkmark \quad \text{Satisfied} \tag{9.51a}$$

$$\frac{\partial \cancelto{0}{N_{xy}}}{\partial x} + \cancelto{0}{\frac{\partial N_y}{\partial y}} = 0 \quad \checkmark \quad \text{Satisfied} \tag{9.51b}$$

$$\frac{\partial V_x}{\partial x} + \cancelto{0}{\frac{\partial V_y}{\partial y}} + q(x) = 0 \quad \Rightarrow \quad KA_{55}\left(\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w_0}{\partial x^2}\right) + q(x) = 0 \tag{9.51c}$$

$$\frac{\partial M_x}{\partial x} + \cancelto{0}{\frac{\partial M_{xy}}{\partial y}} - V_x = 0 \quad \Rightarrow \quad D_{11}\frac{\partial^2 \phi_x}{\partial x^2} - KA_{55}\left(\phi_x + \frac{\partial w_0}{\partial x}\right) = 0 \tag{9.51d}$$

$$\frac{\partial \cancelto{0}{M_{xy}}}{\partial x} + \cancelto{0}{\frac{\partial M_y}{\partial y}} - \cancelto{0}{V_y} = 0 \quad \checkmark \quad \text{Satisfied} \tag{9.51e}$$

Thus, three of the equilibrium equations are identically satisfied. Integrating equation (9.51c) with respect to x gives,

$$\phi_x + \frac{\partial w_0}{\partial x} = -\frac{1}{KA_{55}} \int q(x)\, dx \tag{9.52}$$

Substituting equation (9.52) into equation (9.51d) gives

$$D_{11} \frac{\partial^2 \phi_x}{\partial x^2} = KA_{55} \left(\phi_x + \frac{\partial w_0}{\partial x} \right) = -\int q(x)\, dx \tag{9.53}$$

Equation (9.53) is integrated twice to obtain the rotation ϕ_x. The resulting expression for ϕ_x is substituted into equation (9.52) and integrated once to obtain $w_0(x)$. The integration constants are obtained by enforcing the boundary conditions at $x = 0$ and $x = a$.

EXAMPLE 9.1: Simply supported symmetric cross-ply laminates under uniform loading

In this example, we consider the cylindrical bending of a symmetric cross-ply laminate that is subjected to the following simply supported boundary conditions at the edges $x = 0$ and $x = a$

$$\begin{aligned} w_0 &= 0 \quad \text{at} \quad x = 0, a \\ M_x &= 0 \quad \text{at} \quad x = 0, a \end{aligned} \tag{9.54}$$

The laminate is subjected to uniformly distributed load of magnitude q_0, i.e., $q(x) = q_0$ as shown in Fig. 9.4.

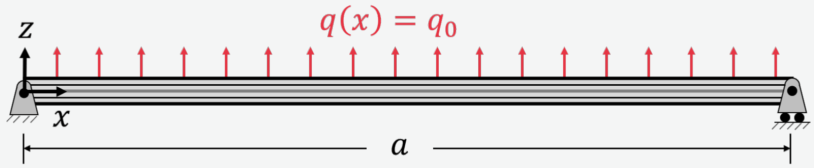


Figure 9.4: Cylindrical bending of a simply supported symmetric cross-ply laminate under a uniform distributed load

In the case of a uniformly distributed load, the right hand side of equation (9.53) can be explicitly integrated to obtain

$$D_{11} \frac{\partial^2 \phi_x}{\partial x^2} = -\int q(x)\, dx = -\int q_0 dx = -q_0 (x + c_1 a) \tag{9.55}$$

Integrating equation (9.55) twice w.r.t. x yields,

$$D_{11}\frac{\partial \phi_x}{\partial x} = -q_0\left(\frac{x^2}{2} + c_1 a x + c_2 a^2\right) \tag{9.56}$$

$$D_{11}\phi_x = -q_0\left(\frac{x^3}{6} + \frac{c_1 a x^2}{2} + c_2 a^2 x + c_3 a^3\right) \tag{9.57}$$

Therefore, it follows that

$$\phi_x(x) = -\frac{q_0}{D_{11}}\left(\frac{x^3}{6} + \frac{c_1 a x^2}{2} + c_2 a^2 x + c_3 a^3\right) \tag{9.58}$$

The deflection w_0 is obtained by integrating the term on the right hand side of (9.52) and substituting for ϕ_x from (9.58) as follows

$$\begin{aligned}
&\phi_x + \frac{\partial w_0}{\partial x} = -\frac{1}{KA_{55}}\int q\,(x)\,dx \\
\Rightarrow\ &\phi_x + \frac{\partial w_0}{\partial x} = -\frac{1}{KA_{55}} q_0\,(x + c_1 a) \\
\Rightarrow\ &\frac{\partial w_0}{\partial x} = \frac{q_0}{D_{11}}\left(\frac{x^3}{6} + \frac{c_1 a x^2}{2} + c_2 a^2 x + c_3 a^3\right) - \frac{1}{KA_{55}} q_0\,(x + c_1 a) \\
\Rightarrow\ &w_0\,(x) = \frac{q_0}{D_{11}}\left(\frac{x^4}{24} + \frac{c_1 a x^3}{6} + \frac{c_2 a^2 x^2}{2} + c_3 a^3 x + c_4 a^4\right) - \frac{1}{KA_{55}} q_0\left(\frac{x^2}{2} + c_1 a x\right)
\end{aligned} \tag{9.59}$$

The integration constants c_1, c_2, c_3 and c_4 are obtained by enforcing the boundary conditions at $x = 0$ and $x = a$.

The bending moment M_x is obtained using (9.48b) and (9.58)

$$M_x = D_{11}\frac{\partial \phi_x}{\partial x} = -q_0\left(\frac{x^2}{2} + c_1 a x + c_2 a^2\right) \tag{9.60}$$

The boundary conditions at $x = 0$ yield

$$\begin{aligned}
w_0\,(0) = 0 \quad &\Rightarrow \quad c_4 = 0 \\
M_x\,(0) = 0 \quad &\Rightarrow \quad c_2 = 0
\end{aligned} \tag{9.61}$$

Substituting for the constants c_2 and c_4 from equation (9.61) into equations (9.59), (9.58) and (9.60) for w_0, M_x and ϕ_x, respectively, yields

$$
\begin{aligned}
w_0(x) &= \frac{q_0}{D_{11}}\left(\frac{x^4}{24}+\frac{c_1 a x^3}{6}+c_3 a^3 x\right)-\frac{q_0}{KA_{55}}\left(\frac{x^2}{2}+c_1 a x\right)\\
\phi_x(x) &= -\frac{q_0}{D_{11}}\left(\frac{x^3}{6}+\frac{c_1 a x^2}{2}+c_3 a^3\right)\\
M_x(x) &= -q_0\left(\frac{x^2}{2}+c_1 a x\right)
\end{aligned}
\tag{9.62}
$$

The boundary conditions at $x = a$ yield

$$
\begin{aligned}
M_x(a) = 0 \quad &\Rightarrow \quad \frac{a^2}{2}+c_1 a^2 = 0\\
&\Rightarrow \quad c_1 = -\frac{1}{2}\\
w_0(a) = 0 \quad &\Rightarrow \quad \frac{q_0}{D_{11}}\left(\frac{a^4}{24}-\frac{a^4}{12}+c_3 a^4\right)-\frac{q_0}{KA_{55}}\left(\frac{a^2}{2}-\frac{a^2}{2}\right)=0\\
&\Rightarrow \quad \frac{q_0 a^4}{D_{11}}\left(\frac{1}{24}-\frac{1}{12}+c_3\right)=0\\
&\Rightarrow \quad c_3 = \frac{1}{24}
\end{aligned}
\tag{9.63}
$$

Substituting the constants c_1 and c_3 from equation (9.63) into Eqn. (9.62) for w_0 yields,

$$
\begin{aligned}
w_0(x) &= \frac{q_0}{D_{11}}\left(\frac{x^4}{24}-\frac{ax^3}{12}+\frac{a^3 x}{24}\right)-\frac{q_0}{KA_{55}}\left(\frac{x^2}{2}-\frac{ax}{2}\right)\\
\Rightarrow w_0(x) &= \frac{q_0}{24D_{11}}\left(x^4-2ax^3+a^3x\right)+\frac{q_0 x}{2KA_{55}}(a-x)
\end{aligned}
\tag{9.64}
$$

Substituting for the constants c_1 and c_3 from equation (9.63) into Eqn. (9.62) for ϕ_x yields,

$$
\phi_x(x) = -\frac{q_0}{D_{11}}\left(\frac{x^3}{6}-\frac{ax^2}{4}+\frac{a^3}{24}\right) = -\frac{q_0}{24D_{11}}\left(4x^3-6ax^2+a^3\right) \tag{9.65}
$$

The maximum vertical defection w_{max} is obtained by evaluating the deflection w_0 at the mid-span $x = a/2$

$$
w_{max} = w_0\left(\frac{a}{2}\right) = \frac{q_0}{24D_{11}}\left(\frac{a^4}{16}-\frac{2a^4}{8}+\frac{a^4}{2}\right)+\frac{q_0}{2KA_{55}}\frac{a}{2}\left(\frac{a}{2}\right) \tag{9.66}
$$

which simplifies to

$$
w_{max} = \underbrace{\frac{5q_0a^4}{384D_{11}}}_{\substack{\text{CLPT prediction}\\ \text{maximum deflection}}} + \underbrace{\frac{q_0a^2}{8KA_{55}}}_{\text{Shear deformation}} \tag{9.67}
$$

The maximum deflection can be written as

$$w_{max} = w'_{max} + \frac{q_0 a^2}{8KA_{55}} \tag{9.68}$$

where $w'_{max} = 5q_0a^4/(384D_{11})$ is the classical laminated plate theory prediction for the maximum deflection.

The maximum deflection w_{max} can be expressed as

$$w_{max} = w'_{max}\,(1+S) \tag{9.69}$$

where the coefficient S captures the contribution of shear deformation to the overall deflection and is defined as

$$S = \frac{q_0a^2}{8KA_{55}} \cdot \frac{1}{w'_{max}} = \frac{q_0a^2}{8KA_{55}} \cdot \frac{384D_{11}}{5q_0a^4} = \frac{48D_{11}}{5KA_{55}a^2} \tag{9.70}$$

In the case of a single orthotropic layer, the bending rigidity D_{11} is

$$D_{11} = \frac{1}{3}Q_{11}\left(z_2^3 - z_1^3\right) = \frac{1}{3}Q_{11}\left[\left(\frac{H}{2}\right)^3 - \left(-\frac{H}{2}\right)^3\right] = \frac{Q_{11}H^3}{12} \tag{9.71}$$

and the shear rigidity A_{55} is

$$A_{55} = \sum_{k=1}^{N}\left(z_{k+1} - z_k\right)\overline{C}_{55}^{(k)} = G_{13}H \tag{9.72}$$

It follows from equations (9.70), (9.71) and (9.72) that for a single orthotropic plate,

$$S = \frac{48D_{11}}{5KA_{55}a^2} = \frac{4Q_{11}H^3}{5K\,(G_{13}H)\,a^2} = \left(\frac{4}{5K}\right)\left(\frac{Q_{11}}{G_{13}}\right)\left(\frac{H}{a}\right)^2 \tag{9.73}$$

The contribution from the shear deformation can be large when,

1. Q_{11}/G_{13} is large, i.e., when the transverse shear modulus is small compared the in-plane Young's modulus

   In the case of a unidirectional IM7-8552 carbon fiber reinforced ply,

$$Q_{11} = 168.44\ \text{GPa},\ G_{13} = G_{12} = 4.80\ \text{GPa} \quad \Rightarrow \quad \frac{Q_{11}}{G_{13}} = 35.1 \tag{9.74}$$

2. H/a is large, i.e., in the case of thick or moderately thick plates. Assuming a shear correction factor of $K = \frac{5}{6}$,

If $\frac{H}{a} = \frac{1}{20}$ then $S = \frac{4}{5K}(35.1)\left(\frac{1}{20}\right)^2 = 0.084$, i.e., a 8.4% increase in the deflection.

If $\frac{H}{a} = \frac{1}{10}$ then $S = \frac{4}{5K}(35.1)\left(\frac{1}{10}\right)^2 = 0.337$, i.e., a 33.7% increase in deflection!

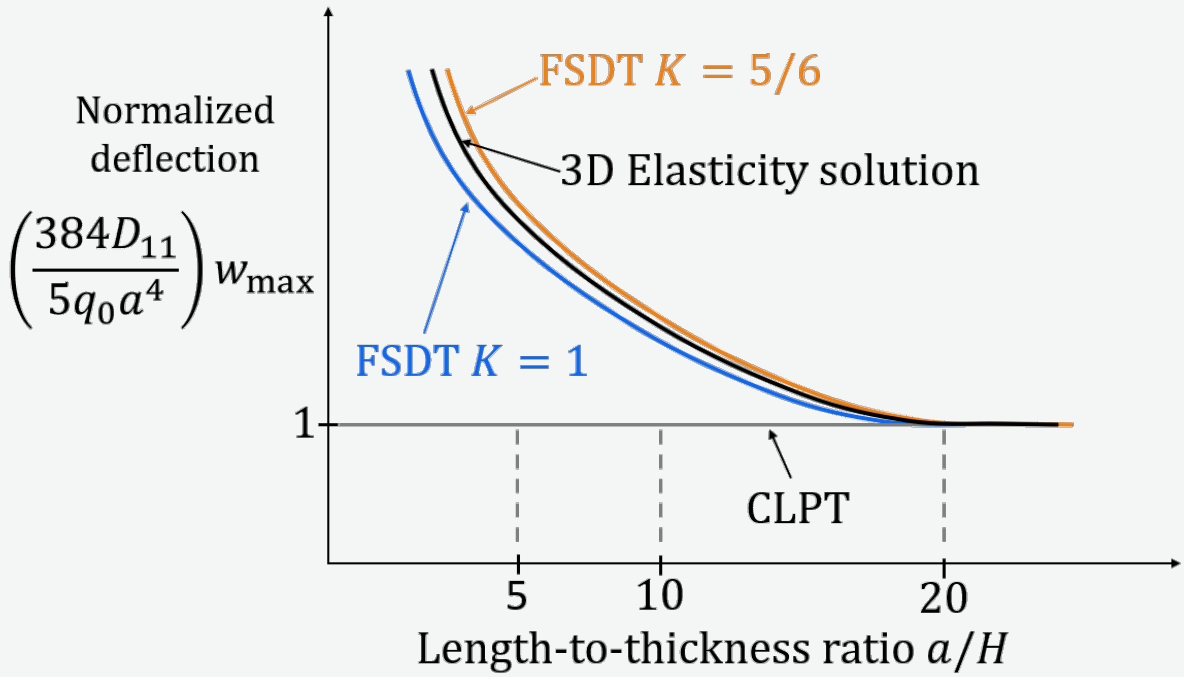


Figure 9.5: Normalized deflection of a laminated plate for varying length to thickness ratios

In the case of an isotropic material,

$$Q_{11} = \frac{E}{1-\nu^2},\ G_{13} = G = \frac{E}{2(1+\nu)} \quad \Rightarrow \quad \frac{Q_{11}}{G_{13}} = \frac{2(1+\nu)}{(1-\nu^2)} = \frac{2}{1-\nu} \tag{9.75}$$

In the case of an Aluminum plate with $\nu = \frac{1}{3}$, $\frac{H}{a} = \frac{1}{10}$ and an assumed value of $K = \frac{5}{6}$,

$$S = \frac{4}{5K}\left(\frac{Q_{11}}{G_{13}}\right)\left(\frac{H}{a}\right)^2 = 0.029 \tag{9.76}$$

i.e., a 2.9% increase in deflection due to transverse shear compared to the 33.7% increase for a unidirectional IM7-8852 plate.

9.5 Free vibration of symmetric cross ply laminates in cylindrical bending

Consider the cylindrical bending vibration of a simply supported cross-ply laminate shown in Fig. 9.6.

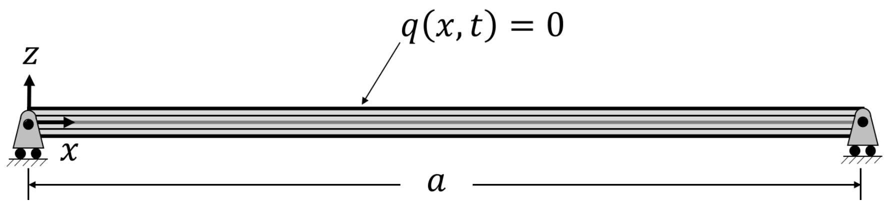


Figure 9.6: Free vibration of a simply supported cross-ply laminate

The boundary conditions at $x = 0$ and $x = a$ are assumed to be,

$$w_0 = 0, \quad M_x = 0, \quad N_x = 0, \quad v_0 = 0, \quad \phi_y = 0 \tag{9.77}$$

In the case of a symmetric cross-ply laminate

$$A_{16} = A_{26} = 0, \quad A_{45} = 0, \quad B_{ij} = 0, \quad D_{16} = D_{26} = 0 \tag{9.78}$$

We assume that

$$\frac{\partial\,(\cdot)}{\partial y} = 0, \quad u_0(x) = 0, \quad v_0(x) = 0, \quad \phi_y(x) = 0 \tag{9.79}$$

The mid-surface deflection $w_0(x,t)$ and rotation $\phi_x(x,t)$ are assumed to have the following forms

$$\begin{aligned} w_0\,(x,t) &= W_m \sin\frac{m\pi x}{a} e^{i\omega t} \\ \phi_x\,(x,t) &= \Phi_m \cos\frac{m\pi x}{a} e^{i\omega t} \end{aligned} \tag{9.80}$$

where the integer m defines the mode shape.

9.5.1 Strains

The strains follow from the definition and Eqn. (9.80) as,

$$\varepsilon_x^0 = 0, \quad \varepsilon_y^0 = 0, \quad \gamma_{xy}^0 = 0 \tag{9.81a}$$

$$\gamma_{yz} = \phi_y + \frac{\partial w_0}{\partial y} = 0 \tag{9.81b}$$

$$\gamma_{xz} = \phi_x + \frac{\partial w_0}{\partial x} = \left[\Phi_m + W_m\left(\frac{m\pi}{a}\right)\right] \cos\frac{m\pi x}{a} e^{i\omega t} \tag{9.81c}$$

The κ's follow from the definitions and and Eqn. (9.80)

$$\kappa_x = \frac{\partial \phi_x}{\partial x} = -\Phi_m \left(\frac{m\pi}{a}\right) \sin \frac{m\pi x}{a} e^{i\omega t} \tag{9.82a}$$

$$\kappa_y = \frac{\partial \phi_y}{\partial y} = 0 \tag{9.82b}$$

$$\kappa_{xy} = \cancelto{0}{\frac{\partial \phi_x}{\partial y}} + \frac{\partial \cancelto{0}{\phi_y}}{\partial x} = 0 \tag{9.82c}$$

9.5.2 Force and moment resultants

The force and moment resultants follow from the definitions (9.14), the strains (9.81) and the κ's (9.82)

$$N_x = N_y = N_{xy} = 0 \tag{9.83a}$$

$$M_x = D_{11}\kappa_x = -D_{11}\Phi_m \left(\frac{m\pi}{a}\right) \sin \frac{m\pi x}{a} e^{i\omega t} \tag{9.83b}$$

$$M_y = D_{12}\kappa_x = -D_{12}\Phi_m \left(\frac{m\pi}{a}\right) \sin \frac{m\pi x}{a} e^{i\omega t} \tag{9.83c}$$

$$M_{xy} = 0 \tag{9.83d}$$

The transverse shear force resultants follow from (9.22) and the transverse shear strains (9.81)

$$\begin{Bmatrix} V_y \\ V_x \end{Bmatrix} = K \begin{bmatrix} A_{44} & \cancelto{0}{A_{45}} \\ \cancelto{0}{A_{45}} & A_{55} \end{bmatrix} \cdot \begin{Bmatrix} \cancelto{0}{\gamma_{yz}} \\ \gamma_{xz} \end{Bmatrix} \tag{9.84}$$

It follows from equation (9.84) that

$$V_y = 0 \tag{9.85a}$$

$$V_x = KA_{55}\gamma_{xz} = KA_{55} \left[\Phi_m + W_m \left(\frac{m\pi}{a}\right)\right] \cos \frac{m\pi x}{a} e^{i\omega t} \tag{9.85b}$$

9.5.3 Equations of motion

The equation of motion (9.37a) is identically satisfied since

$$\frac{\partial \cancelto{0}{N_x}}{\partial x} + \frac{\partial \cancelto{0}{N_{xy}}}{\partial y} = I_0 \frac{\partial^2 \cancelto{0}{u_0}}{\partial t^2} + \cancelto{0}{I_1} \frac{\partial^2 \phi_x}{\partial t^2} \quad \checkmark \text{ Satisfied} \tag{9.86a}$$

Similarly, the equation of motion (9.37b) is identically satisfied

$$\frac{\partial \cancelto{0}{N_{xy}}}{\partial x} + \cancelto{0}{\frac{\partial N_y}{\partial y}} = I_0 \frac{\partial^2 \cancelto{0}{v_0}}{\partial t^2} + \cancelto{0}{I_1} \frac{\partial^2 \phi_y}{\partial t^2} \quad \checkmark \text{ Satisfied} \tag{9.86b}$$

The equation of motion (9.37c) reduces to

$$\begin{aligned} \frac{\partial V_x}{\partial x} + \cancelto{0}{\frac{\partial V_y}{\partial y}} + \cancelto{0}{q(x,t)} &= I_0 \frac{\partial^2 w_0}{\partial t^2} \\ \Rightarrow -KA_{55}\left[\Phi_m + W_m\left(\frac{m\pi}{a}\right)\right]\left(\frac{m\pi}{a}\right)\sin\frac{m\pi x}{a} e^{i\omega t} &= -I_0 W_m \omega^2 \sin\frac{m\pi x}{a} e^{i\omega t} \\ \Rightarrow KA_{55}\left(\frac{m\pi}{a}\right)\Phi_m + \left(KA_{55}\frac{m^2\pi^2}{a^2} - I_0\omega^2\right) W_m &= 0 \end{aligned} \tag{9.86c}$$

The equation of motion (9.37d) reduces to

$$\begin{aligned} \frac{\partial M_x}{\partial x} + \frac{\partial \cancelto{0}{M_{xy}}}{\partial y} - V_x &= \cancelto{0}{I_1} \frac{\partial^2 u_0}{\partial t^2} + I_2 \frac{\partial^2 \phi_x}{\partial t^2} \\ \Rightarrow -D_{11}\Phi_m\left(\frac{m\pi}{a}\right)^2 \cos\frac{m\pi x}{a} e^{i\omega t} - KA_{55}\left[\Phi_m + W_m\left(\frac{m\pi}{a}\right)\right]\cos\frac{m\pi x}{a} e^{i\omega t} &= \\ - I_2\Phi_m\omega^2 \cos\frac{m\pi x}{a} e^{i\omega t} & \\ \Rightarrow \left(D_{11}\frac{m^2\pi^2}{a^2} + KA_{55} - I_2\omega^2\right)\Phi_m + KA_{55}\left(\frac{m\pi}{a}\right) W_m &= 0 \end{aligned} \tag{9.86d}$$

The equation of motion (9.37e) is identically satisfied

$$\frac{\partial \cancelto{0}{M_{xy}}}{\partial x} + \cancelto{0}{\frac{\partial M_y}{\partial y}} - \cancelto{0}{V_y} = \cancelto{0}{I_1} \frac{\partial^2 v_0}{\partial t^2} + I_2 \frac{\partial^2 \cancelto{0}{\phi_y}}{\partial t^2} \quad \checkmark \text{ Satisfied} \tag{9.86e}$$

Equations (9.86c) and (9.86d) can be written in matrix form as,

$$\begin{bmatrix} \left(D_{11}\frac{m^2\pi^2}{a^2} + KA_{55} - I_2\omega^2\right) & KA_{55}\left(\frac{m\pi}{a}\right) \\ KA_{55}\left(\frac{m\pi}{a}\right) & \left(KA_{55}\frac{m^2\pi^2}{a^2} - I_0\omega^2\right) \end{bmatrix} \cdot \begin{Bmatrix} \Phi_m \\ W_m \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{9.87}$$

For a non-tirivial solution, the determinant of the matrix in (9.87) must vanish. Therefore,

$$
\begin{aligned}
KA_{55}\frac{m^2\pi^2}{a^2}\left(D_{11}\frac{m^2\pi^2}{a^2}+\cancel{KA_{55}}\right)-\left(I_0D_{11}\frac{m^2\pi^2}{a^2}+I_0KA_{55}+I_2KA_{55}\frac{m^2\pi^2}{a^2}\right)\omega^2 \\
+I_0I_2\omega^4-\cancel{K^2A_{55}^2\frac{m^2\pi^2}{a^2}}=0
\end{aligned}
\tag{9.88}
$$

Eqn. (9.88) simplifies to,

$$
I_0I_2\omega^4-\left(I_0D_{11}\frac{m^2\pi^2}{a^2}+I_0KA_{55}+I_2KA_{55}\frac{m^2\pi^2}{a^2}\right)\omega^2+KD_{11}A_{55}\frac{m^4\pi^4}{a^4}=0 \tag{9.89}
$$

which can be expressed as

$$
\omega^4-\left(\frac{D_{11}}{I_2}\frac{m^2\pi^2}{a^2}+\frac{K}{I_2}A_{55}+\frac{K}{I_0}A_{55}\frac{m^2\pi^2}{a^2}\right)\omega^2+K\frac{D_{11}A_{55}}{I_0I_2}\frac{m^4\pi^4}{a^4}=0 \tag{9.90}
$$

It is possible to solve Eqn. (9.90) for the natural frequency ω_m with shear deformation and rotary inertia included.

If the rotary inertia is neglected by setting $I_2=0$ in Eqn. (9.89), we obtain

$$
\omega_m^2=\frac{KD_{11}A_{55}\frac{m^4\pi^4}{a^4}}{I_0\left(D_{11}\frac{m^2\pi^2}{a^2}+KA_{55}\right)} \tag{9.91}
$$

The natural frequency ω_m follows from equation (9.91) as,

$$
\omega_m=\frac{m^2\pi^2}{a^2}\sqrt{\frac{D_{11}}{I_0}}\sqrt{\frac{KA_{55}}{D_{11}\frac{m^2\pi^2}{a^2}+KA_{55}}}=\omega_m'\sqrt{\frac{1}{1+\tilde{S}}} \tag{9.92}
$$

where,

$$
\omega_m'=\frac{m^2\pi^2}{a^2}\sqrt{\frac{D_{11}}{I_0}} \tag{9.93}
$$

is the natural frequency obtained using the classical laminated plate theory, i.e., no shear deformations, and,

$$
\tilde{S}=\frac{D_{11}m^2\pi^2}{KA_{55}a^2} \tag{9.94}
$$

is the shear deformation factor. $\tilde{S}$ captures the effect of transverse shear deformations on the natural frequencies. In the case of a single orthotropic layer,

$$
D_{11}=\frac{Q_{11}H^3}{12},\quad A_{55}=G_{13}H \tag{9.95}
$$

Therefore, it follows from equations (9.95) and (9.94) that,

$$\tilde{S} = \frac{D_{11}m^2\pi^2}{KA_{55}a^2} = \frac{Q_{11}H^3m^2\pi^2}{12KG_{13}Ha^2} = \frac{Q_{11}H^2m^2\pi^2}{12KG_{13}a^2} = \frac{\pi^2}{12K}\left(\frac{Q_{11}}{G_{13}}\right)\underbrace{\left(\frac{H}{a}\right)^2 m^2}_{\left(\frac{mH}{a}\right)^2} \tag{9.96}$$

The shear deformation factor $\tilde{S}$ increases as Q_{11}/G_{13}, H/a or m increase, i.e., if G_{13} is small and/or the plate is moderately thick and/or higher modes. As S increases, the natural frequency ω_m decreases, i.e., shear deformation decreases the natural frequency as shown in Fig. 9.7.

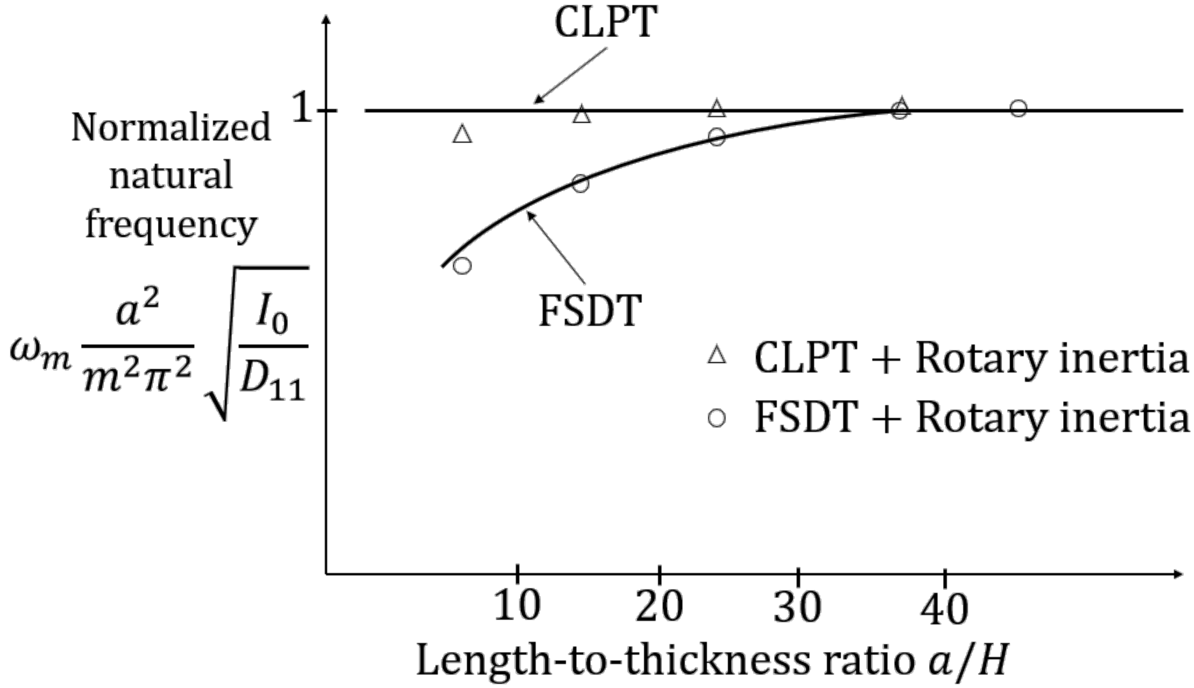


Figure 9.7: Normalized natural frequency for varying length to thickness ratio

9.6 Navier solution for cross ply laminates

Consider the bending of cross-ply rectangular laminates that are simply supported on all edges.

For a rectangular cross-ply laminate,

$$A_{16} = A_{26} = 0, \quad B_{16} = B_{26} = 0, \quad D_{16} = D_{26} = 0, \quad A_{45} = 0 \tag{9.97}$$

The boundary conditions at $x = 0$ and $x = a$ are,

$$w_0 = 0, \quad M_x = 0, \quad N_x = 0, \quad v_0 = 0, \quad \phi_y = 0 \tag{9.98}$$

The boundary conditions at $y = 0$ and $y = b$ are,

$$w_0 = 0, \quad M_y = 0, \quad N_y = 0, \quad u_0 = 0, \quad \phi_x = 0 \tag{9.99}$$

The Navier solution for the plate bending is,

The loads $q(x, y)$ are expanded as a double Fourier series as,

$$q(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Q_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{9.100a}$$

The displacement u_0 is expanded as,

$$u_0 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} U_{mn} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{9.100b}$$

The displacement v_0 is expanded as,

$$v_0 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} V_{mn} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \tag{9.100c}$$

The displacement w_0 is expanded as,

$$w_0 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{9.100d}$$

The rotation ϕ_x is expanded as,

$$\phi_x = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} X_{mn} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{9.100e}$$

The rotation ϕ_y is expanded as,

$$\phi_y = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \tag{9.100f}$$

As with the classical laminated plate theory, substituting equations (9.100) into the equilibrium equations will yield a system of equations for U_{mn}, V_{mn}, W_{mn}, X_{mn} and Y_{mn},

$$[K]_{5\times5}\begin{Bmatrix} U_{mn} \\ V_{mn} \\ W_{mn} \\ X_{mn} \\ Y_{mn} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ Q_{mn} \\ 0 \\ 0 \end{Bmatrix} \tag{9.101}$$

The values of U_{mn}, V_{mn}, W_{mn}, X_{mn} and Y_{mn} can be calculated for each m and n.

Sandwich Composites 10

Sandwich composites are a class of composite materials that consist of two thin laminated face sheets that are separated by a thick layer of a less dense material known as the core. The face sheets bear most of the bending loads while the core supports the transverse shear force as shown in Fig. 10.1.

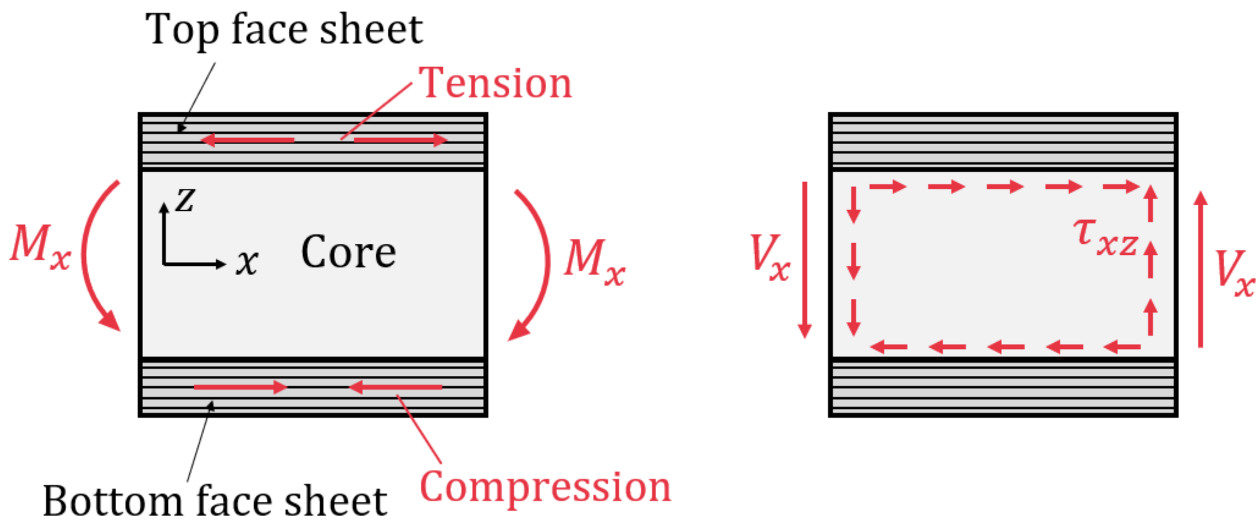


Figure 10.1: Stresses in the face sheets and core of a sandwich composite

10.1 Geometry and representative properties

10.1.1 Geometry of sandwich composites

The top and bottom face sheets consists of multiple laminae each of thickness h. The number of laminae in the top and bottom face sheets are N_t and N_b, respectively. The thickness of the top and face sheets are,

$$\begin{aligned} H_t &= N_t \cdot h \\ H_b &= N_b \cdot h \end{aligned} \tag{10.1}$$

where H_t is the thickness of the top face sheet and H_b is the thickness of the bottom face sheet. The core is of thickness H_c.

The total number of layers, including the core, is

$$N = N_b + N_t + 1 \tag{10.2}$$

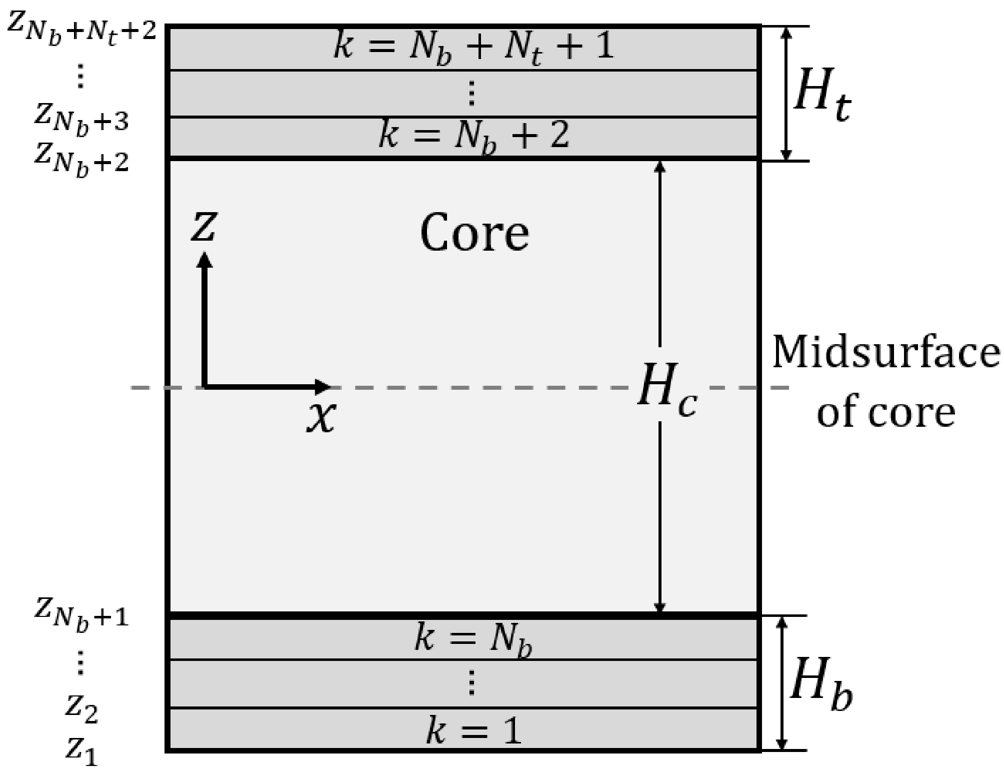


Figure 10.2: Sandwich composite layer numbering and interface locations

The layers are numbered from bottom to top with the core numbered $N_b + 1$. The orientations of the plies are denoted by θ_k with k varying from 1 to $N_b + N_t + 1$.

The midsurface of the core is taken as the reference surface $z = 0$. The locations of the interfaces are as follows:

Bottom facing $(1 \leq k \leq N_b + 1)$

$$z_k = -\frac{H_c}{2} - H_b + (k-1)\,h \tag{10.3}$$

Note that the z-coordinate of the bottom surface of the core is

$$z_{N_b+1} = -\frac{H_c}{2} - H_b + \cancelto{H_b}{N_b h} = -\frac{H_c}{2} \tag{10.4}$$

as expected.

Top facing $(N_b + 2 \leq k \leq N_b + N_t + 2)$

$$z_k = \frac{H_c}{2} + (k - N_b - 2)\,h \tag{10.5}$$

Note that the z-coordinate of the top surface of the laminate is

$$z_{N_b+N_t+2} = \frac{H_c}{2} + \left(\cancel{N_b} + N_t + \cancel{2} - \cancel{N_b} - \cancel{2}\right) h = \frac{H_c}{2} + N_t h = \frac{H_c}{2} + H_t \tag{10.6}$$

as expected.

10.1.2 Representative properties of core materials

The representative density, transverse shear modulus and shear strength for different core materials are listed below.

(a) Balsa (CK57)

Density $\rho = 150\ \mathrm{kg/m^3}$
Shear modulus $G_c = 58.7$ MPa
Shear strength $F_{sc} = 3.7$ MPa

(b) Divinycell foam (H100)

Density $\rho = 100\ \mathrm{kg/m^3}$
Shear modulus $G_c = 50$ MPa
Shear strength $F_{sc} = 1.8$ MPa

(c) Honeycomb core

Honeycomb cores have different shear moduli in the 1-3 and 2-3 planes where the 1- or L-direction is the ribbon direction and the 2- or W-direction is the transverse direction.

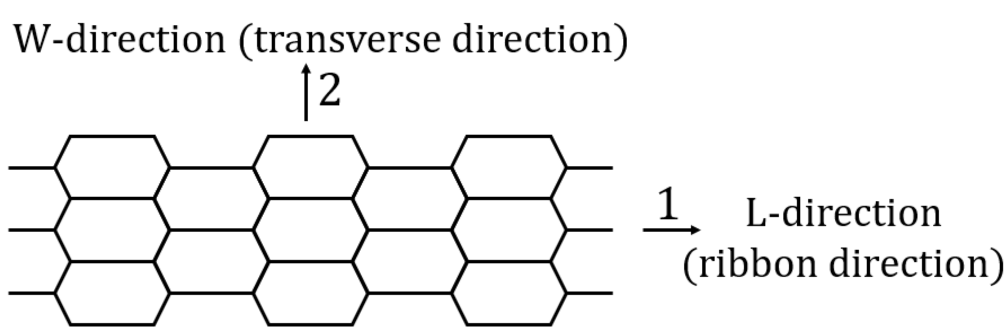


Figure 10.3: Honeycomb core

The properties of Hexcel HRP/F50-45 fiberglass cloth reinforced phenolic resin are,
Density $\rho = 72\ \mathrm{kg/m^3}$
Shear modulus in the L-direction $G_{13} = 220$ MPa
Shear modulus in the W-direction $G_{23} = 90$ MPa

For reference, the density of carbon fiber reinforced composites is approximately 1600 $\mathrm{kg/m^3}$.

10.2 Analysis of sandwich composites

Sandwich composites exhibit significant shear deformation due to their low shear rigidity and high bending rigidity. Therefore, we use the first order shear deformation to analyze sandwich composite plates.

10.2.1 Assumptions

The assumptions used for the analysis of sandwich composites are,

1. The face sheets are assumed to be made of the same material with identical properties and ply thickness h. The plies are assumed state of plane stress, i.e.,

$$\sigma_z = 0, \quad \tau_{xz} = \tau_{yz} = 0 \tag{10.7}$$

2. The core is assumed to have transverse shear moduli G_{xz} and G_{yz} in the $x - z$ and $y - z$ planes respectively. The in-plane elastic moduli and stresses in the core are assumed to be negligible, i.e.,

$$\sigma_x^{(c)} = \sigma_y^{(c)} = \tau_{xy}^{(c)} = 0, \quad E_1^{(c)} = E_2^{(c)} = G_{12}^{(c)} = 0 \tag{10.8}$$

10.2.2 Analysis of sandwich composites

Sandwich composites are analyzed using the first order shear deformation theory. The displacements in the FSDT are assumed to be

$$\begin{aligned} u &= u_0\,(x, y) + z\phi_x\ \ (x, y) \\ v &= v_0\,(x, y) + z\phi_y\ \ (x, y) \\ w &= w_0\,(x, y) \end{aligned} \tag{10.9}$$

where u_0, v_0 and w_0 are the mid-surface displacements, and ϕ_x and ϕ_y are the rotations of the normals in the $x - z$ and $y - z$ planes, respectively. The through-thickness variation of the displacement u is shown in Fig. 10.4.

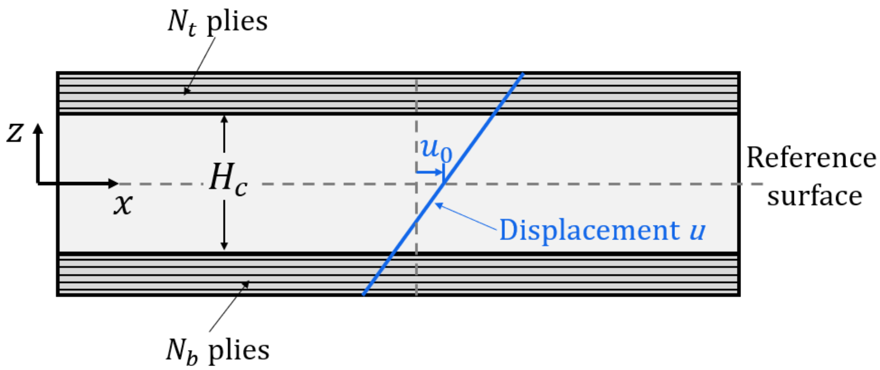


Figure 10.4: Kinematics of a sandwich composite

10.2.3 In-plane and bending rigidities

In order to analyze a sandwich composite plate using FSDT, we first need to calculate the ABD matrix of laminate rigidiites and the transverse shear rigidities A_{44}, A_{55} and A_{45}.

The $[\overline{Q}]$ for the facing plies are calculated as before. The contribution of the core to the bending rigidities of sandwich composites is neglected by setting $[\overline{Q}]$ of the core is set equal to zero, i.e.,

$$\left[\overline{Q}\right]^{(N_b+1)} = [0] \tag{10.10}$$

Therefore, the A, B and D matrices are evaluated as follows

$$\begin{aligned}
[A] &= \sum_{k=1}^{N_b} (z_{k+1} - z_k) \left[\overline{Q}\right]^{(k)} + \sum_{k=N_b+2}^{N_b+N_t+1} (z_{k+1} - z_k) \left[\overline{Q}\right]^{(k)} \\
[B] &= \frac{1}{2} \sum_{k=1}^{N_b} \left(z_{k+1}^2 - z_k^2\right) \left[\overline{Q}\right]^{(k)} + \frac{1}{2} \sum_{k=N_b+2}^{N_b+N_t+1} \left(z_{k+1}^2 - z_k^2\right) \left[\overline{Q}\right]^{(k)} \\
[D] &= \frac{1}{3} \sum_{k=1}^{N_b} \left(z_{k+1}^3 - z_k^3\right) \left[\overline{Q}\right]^{(k)} + \frac{1}{3} \sum_{k=N_b+2}^{N_b+N_t+1} \left(z_{k+1}^3 - z_k^3\right) \left[\overline{Q}\right]^{(k)}
\end{aligned} \tag{10.11}$$

10.2.4 Transverse shear rigidities and shear force resultants

Shear correction factors are not used for sandwich composites since the shear stress and shear strain are fairly constant throughout the thickness of the core as shown in Fig. 10.5. In other words, the shear correction factor

$$K = 1 \tag{10.12}$$

Furthermore, the transverse shear stresses in the facings are assumed to be negligible. The transverse

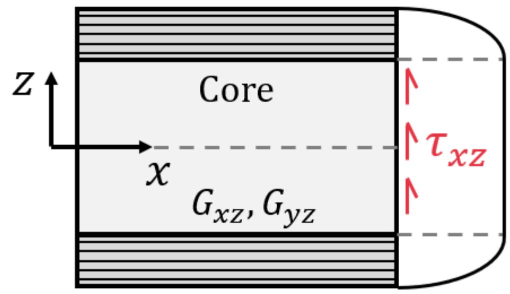


Figure 10.5: Transverse shear stress distribution in a sandwich composite

shear stresses in the core are

$$\begin{Bmatrix} \tau_{yz} \\ \tau_{xz} \end{Bmatrix} = \begin{bmatrix} G_{yz} & 0 \\ 0 & G_{xz} \end{bmatrix} \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} \tag{10.13}$$

where G_{xz} and G_{yz} are the transverse shear modului of the core in the $x-z$ and $y-z$ planes, respectively. It follows from equation (10.13) that the transverse shear force resultants

$$\begin{Bmatrix} V_y \\ V_x \end{Bmatrix} = \int_{-H_c/2}^{H_c/2} \begin{Bmatrix} \tau_{yz} \\ \tau_{xz} \end{Bmatrix} dz = \begin{bmatrix} A_{44} & 0 \\ 0 & A_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} \tag{10.14}$$

where

$$\begin{aligned} A_{44} &= H_c G_{yz} \\ A_{55} &= H_c G_{xz} \end{aligned} \tag{10.15}$$

Note that the transverse shear rigidity $A_{45} = 0$ since it is assumed that the principal material directions of the core are parallel to the x or y directions, i.e. $\theta^{(N_b+1)}$ equals 0° or 90°.

Once we have the rigidities $[A]$, $[B]$, $[D]$, A_{44} and A_{55}, we can use the first order shear deformation theory to analyze sandwich composites by setting the shear correction factor $K = 1$.

EXAMPLE 10.1: Cylindrical bending of a symmetric cross-ply sandwich composite plate

Consider the cylindrical bending of a symmetric cross-ply sandwich composite that is simply supported and subjected to a uniformly distributed load as shown in Fig. 10.6.

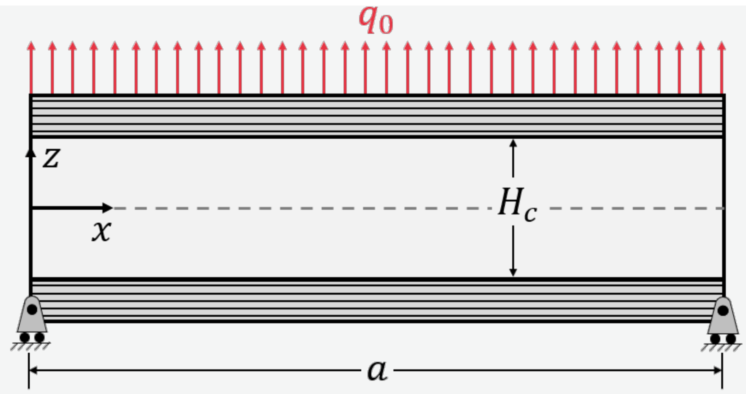


Figure 10.6: Simply supported sandwich composite subjected to a uniform distributed load

An example of a symmetric cross-ply sandwich composite is: $[\underbrace{0/90/0/}_{\text{Bottom}}\text{core}\underbrace{/0/90/0}_{\text{Top}}]$.

In the case of a symmetric sandwich composite with cross-ply laminated face sheets,

$$[B] = [0], \quad A_{16} = A_{26} = 0, \quad D_{16} = D_{26} = 0 \tag{10.16}$$

The transverse shear rigidities are

$$A_{44} = H_c G_{yz}, \quad A_{55} = H_c G_{xz}, \quad A_{45} = 0 \tag{10.17}$$

The same solution process used earlier to analyze the bending problem using the first order shear deformation theory is used for the analysis of sandwich composite plates. In the case of a simply supported sandwich composite plate in cylindrical bending, the maximum deflection for a uniformly distributed load is

$$w_{max} = \frac{5q_0a^4}{384D_{11}} + \frac{q_0a^2}{8KA_{55}} = \frac{5q_0a^4}{384D_{11}} + \frac{q_0a^2}{8A_{55}} \tag{10.18}$$

where the shear correction factor $K = 1$.

Appendix

A

Abaqus Tutorial

A.1 Problem Description

Consider the cross-ply asymmetric laminated square plate shown in Fig. A.1 of length $a = 0.3$ m in the x-direction and width $b = 0.3$ m in the y-direction. The laminated plate is made of IM7/8552 unidirectional carbon fiber-reinforced plies whose properties are listed in Sec. 1.10.1. The laminate has a stacking sequence of $[0_2/90_2]$ with plies of thickness $h = 0.2$ mm each. The laminated plate is subjected to a concentrated point load of magnitude $P = 1N$ acting vertically downward at $x = a/4 = 0.075$ m and $y = b/2 = 0.15$ m.

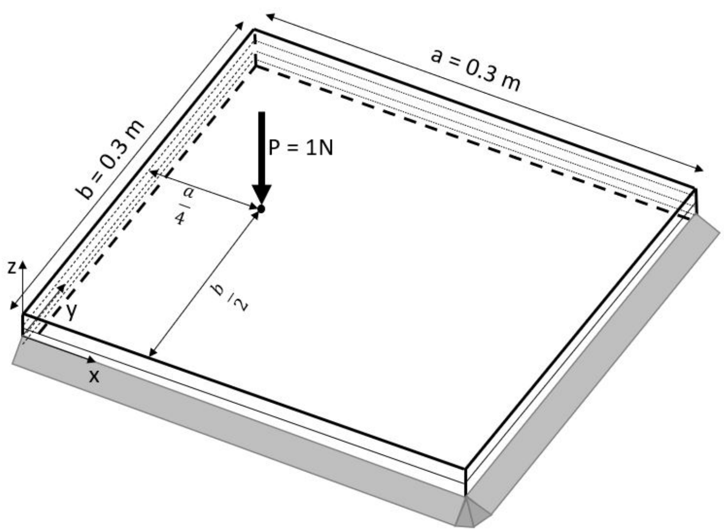


Figure A.1: $[0_2/90_2]$ laminate under a point load

The laminated plates is supported by S_2 simply supported on all edges, i.e.,

$$
\begin{aligned}
w_0 = 0\,,\ M_x = 0\,,\ N_x = 0\,,\ v_0 = 0 \quad \text{at} \quad x = 0,\ a \\
w_0 = 0\,,\ M_y = 0\,,\ u_0 = 0\,,\ N_y = 0 \quad \text{at} \quad y = 0,\ b
\end{aligned}
\tag{A.1}
$$

We are interested in evaluating the following,

1. the maximum vertical deflection of the plate
2. through-thickness variation of stresses at the center of the plate
3. through-thickness variation of the failure index at the center of the plate

The finite element analysis of the laminated composite plate will be performed using Abaqus. The finite element model is created using the Abaqus/CAE (Complete Abaqus Environment) graphical user interface. The preprocessing, simulation and postprocessing are performed using modules as described in the following sections.

NOTES

- Abaqus does not have any default or built-in system of units. Make sure you use *consistent units* to specify input data. In this tutorial, we will use SI units of kg, m and s. Accordingly, the mass density is specified in kg/m^3, force in N, elastic moduli and stresses in Pa and energy in J.

A.2 Abaqus/CAE startup

After starting Abaqus/CAE, the **Start Session** dialog box will appear. The **Create Model Database** startup option will allow you to begin a new analysis. Select **With Standard/Explicit Model** as shown in Fig. A.2.

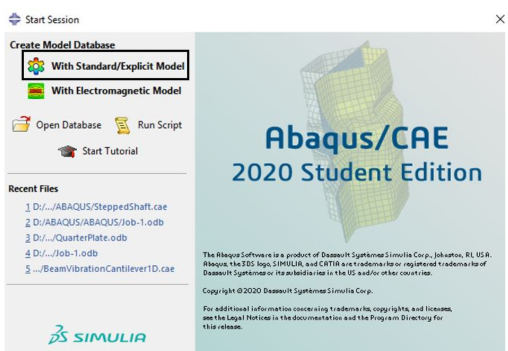


Figure A.2: Abaqus startup screen

That will open the main window as shown in Fig. A.3.

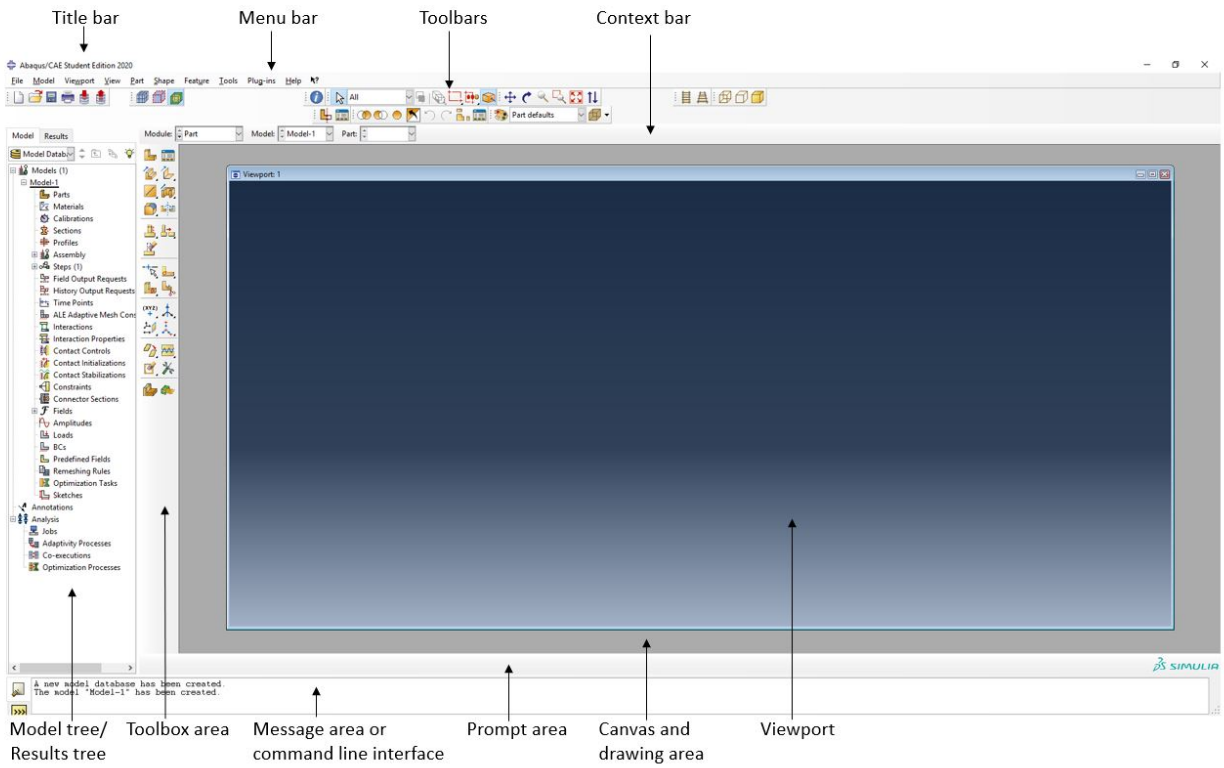


Figure A.3: Abaqus main window

Click **File** → **Set Work Directory** to choose the work directory to save the model and output files as shown in Fig. A.4.

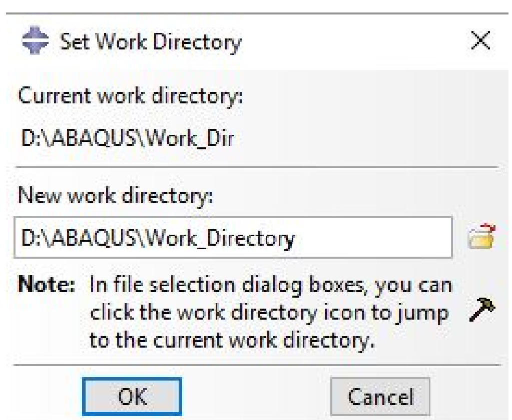


Figure A.4: Setting the work directory

Click **File** → **Save** to save the model. You will be asked to enter a file name when you save the model for the first time as shown in Fig. A.5.

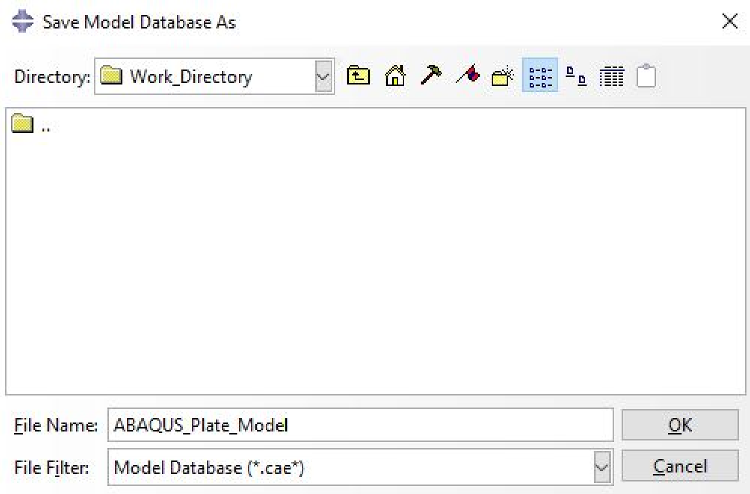


Figure A.5: Saving the model

It is recommended that you save the model periodically so that you don't lose your work.

Abaqus/CAE is divided into units called modules with each module containing only those tools that are relevant to the specific portion of the modeling task. You can select a module from the **Module** list as shown in Fig. A.6.

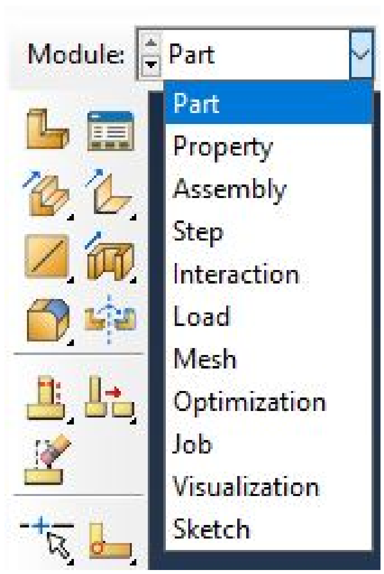


Figure A.6: Abaqus modules

A.3 Module: Part

Upon startup, Abaqus/CAE enters the Part module by default. In the Part module, you will find the toolbox area with a set of icons. Click on the  icon to open the **Create Part** dialog box shown in Fig. A.7.

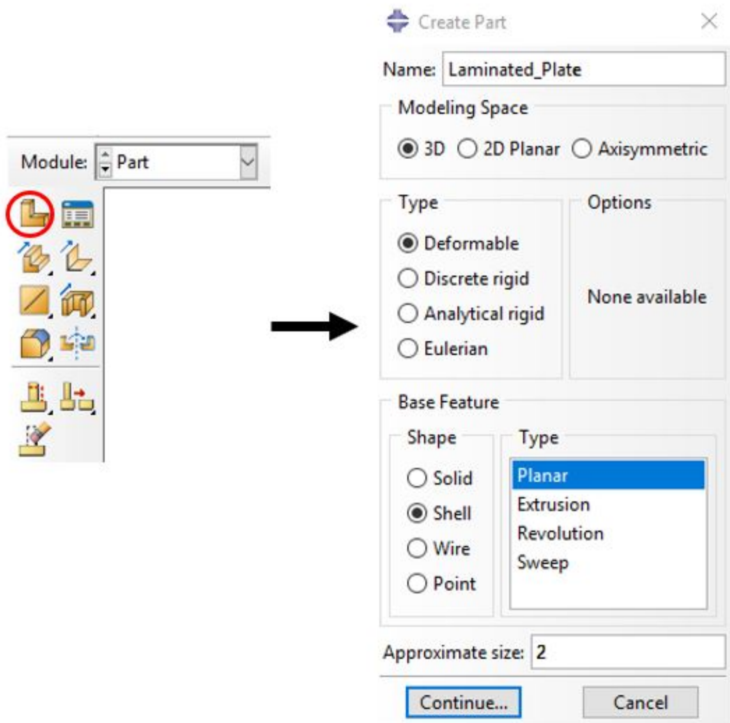


Figure A.7: Create Part dialog box

In the **Create Part** dialog box,

- Name the part, e.g., ***Laminated_Plate***
- Choose the ***3D***, ***Deformable***, ***Shell*** and ***Planar*** options
- In the **Approximate size** text field, type **2.0** for the size of the drawing canvas so that is bigger than the largest dimension of the plate which is 0.3 m
- Click **Continue** to exit the **Create Part** dialog box

Abaqus/CAE enters the Sketcher and creates a Sketcher grid. Click on the  and draw a rectangle or a square as shown in Fig. A.8.

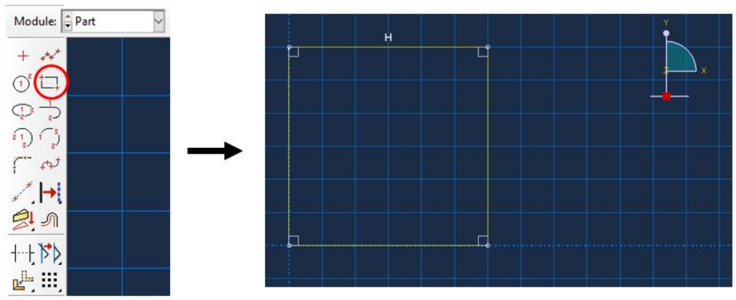


Figure A.8: Drawing a rectangle

After drawing the rectangle, use the **Add Dimension** tool  to dimension the width and height of the rectangle. Click on the top or bottom horizontal edge and enter a new dimension of 0.3 m in the prompt area. Click on the left or right vertical edge and enter a new dimension of 0.3 m in the prompt area. The dimensions of the rectangle should match the dimensions of the plate as shown in Fig. A.9.

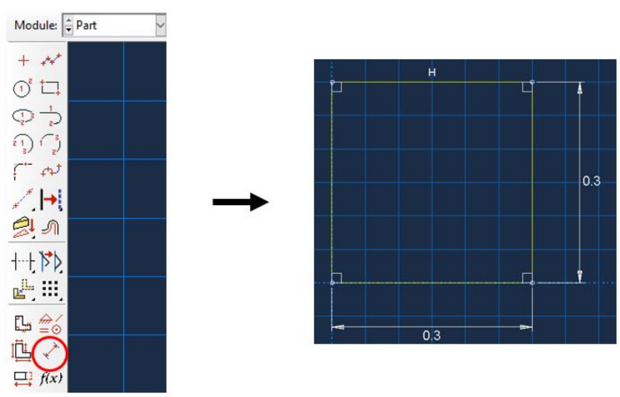


Figure A.9: Dimensioning the rectangle

After dimensioning the sketch, click the **Esc** key on your keyboard then click on **Done** in the prompt area to exit the Sketcher. You should now see a rectangular part in the **Part** module as shown in Fig. A.10. This represents the mid-surface of the laminated composite plate.

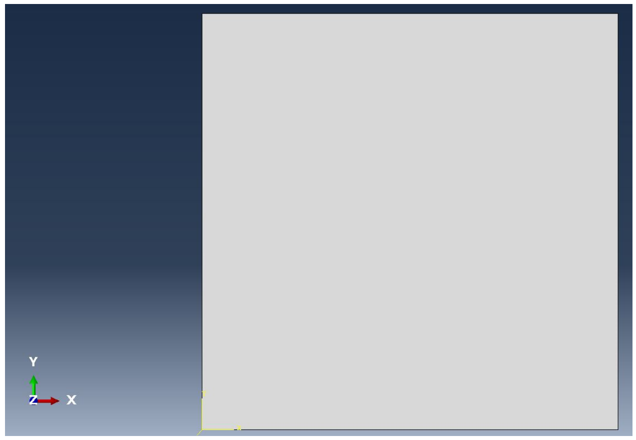


Figure A.10: Rectangular part representing laminate mid-surface

A.4 Module: Property

In this module the properties of the plies will be entered into Abaqus and the composite layup, or ply orientations, created. Switch from the **Part** to the **Property** module in the Context bar.

A.4.1 Defining material properties

To create a material corresponding to the fiber-reinforced plies, under the **Property** module, click on the **Create Material** tool  icon and the **Edit Material** dialog box appears as shown in A.11. Name the material, e.g., ***IM7_8552***.

We will use the properties of IM7/8552 unidirectional carbon fiber-reinforced laminae listed in Sec. 1.10.1. Enter the ply mass density by clicking on **General** → **Mass Density**. The density is not needed for static problems, but needs to be specified for vibration and dynamic problems.

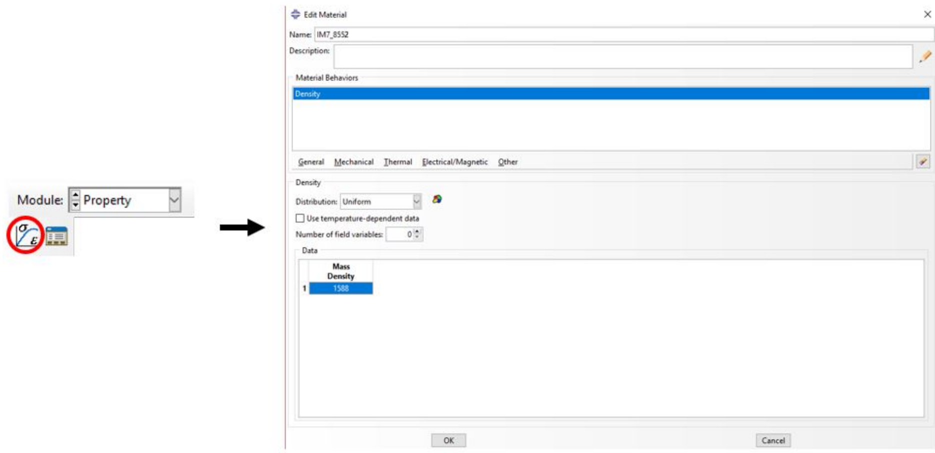


Figure A.11: Mass density

Next, click on **Mechanical** → **Elasticity** → **Elastic** to enter the elastic properties of the laminae. In the **Type** drop-down menu choose **Lamina** as shown in Fig. A.12. Enter the lamina properties E_1, E_2, ν_{12}, G_{12}. The transverse shear moduli G_{13}, G_{23} also need to be specified although they do not have a significantly influence on the response of thin laminates. They are necessary to capture the transverse shear deformation effects for moderately thick laminates. You can enter approximate values for G_{13}, G_{23} if they unknown.

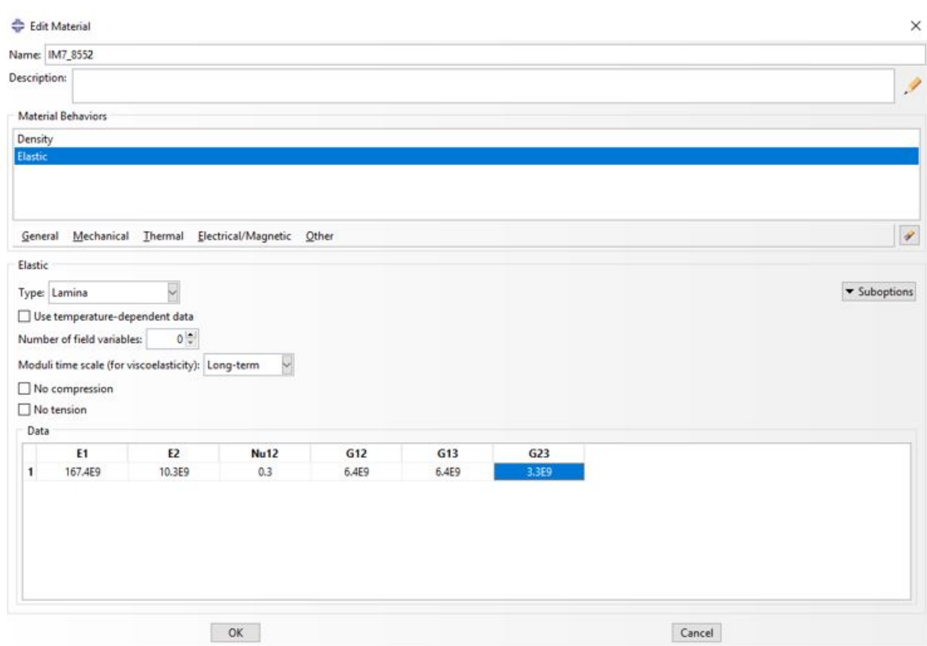


Figure A.12: Lamina elastic properties

In order to perform failure analysis the material strength data needs to be entered. Select the **Suboptions** menu on the far right of **Type: Lamina** and choose **Fail Stress**. In the **Suboption Editor** dialog box that opens, enter the tensile strength F_{1t} in the fiber direction, the compressive strength F_{1c} in the fiber direction, the tensile strength F_{2t} in the transverse direction, the compressive strength F_{2c} in the transverse direction and the shear strength F_6 as shown in Fig. A.13. In the case of Tsai-Wu failure theory, enter the cross product coefficient $f_{12}/\sqrt{f_{11}f_{22}}$. In the absence of biaxial experimental data, a value of -0.5 is normally assumed for the cross product coefficient. If the equibiaxial strength of the lamina is known, the cross product coefficient can be left blank and the biaxial stress limit σ_{biax} entered in the last column.

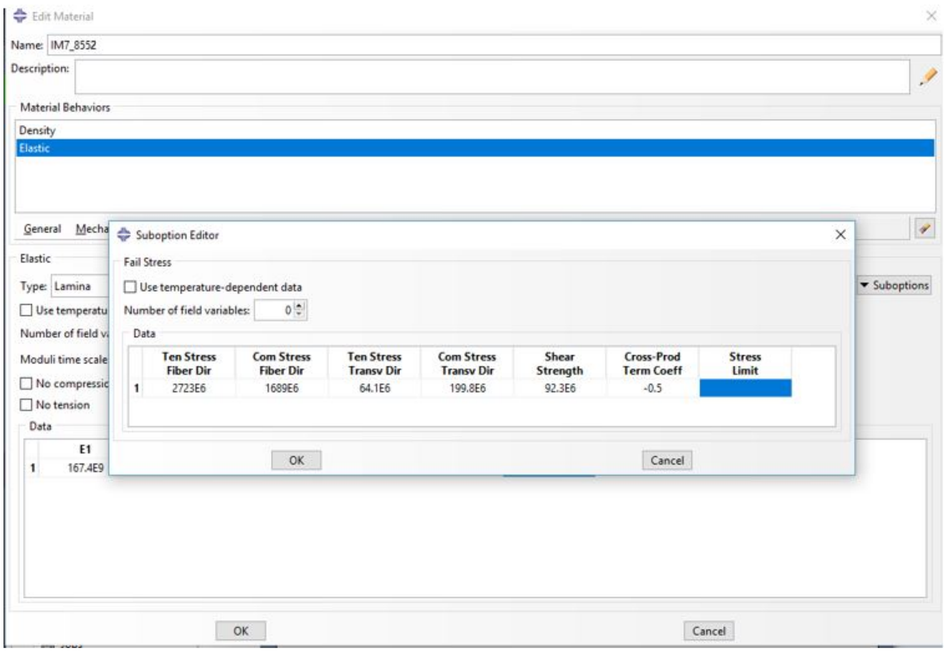
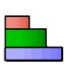


Figure A.13: Specifying failure properties

Click **OK** on all the dialog boxes to save the material properties.

A.4.2 Define the composite layup

Click on the  icon in the **Property** module to create the composite layup. The **Create Composite Layup** dialog box shown in Fig. A.14 will open. Name the layup, e.g., ***Laminate_Lay_up***, specify an ***Initial ply count*** of **4**, choose ***Conventional Shell*** for Element Type and click **Continue**.

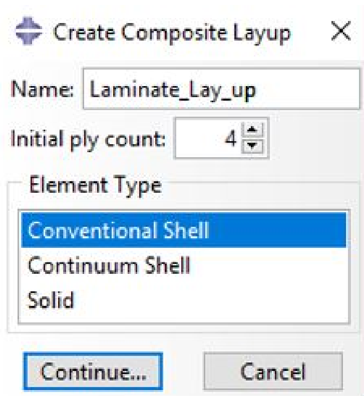


Figure A.14: Create composite layup layup dialog box

In the **Edit Composite Layup** dialog box that appears next we can specify the stacking sequence, or the orientation of the plies. First a global/datum coordinate system needs to be specified. Under **Layup Orientation**, click on the drop-down menu under **Definition** and choose **Coordinate System**. Next click on as shown in Fig. A.15 to create datum coordinate system.

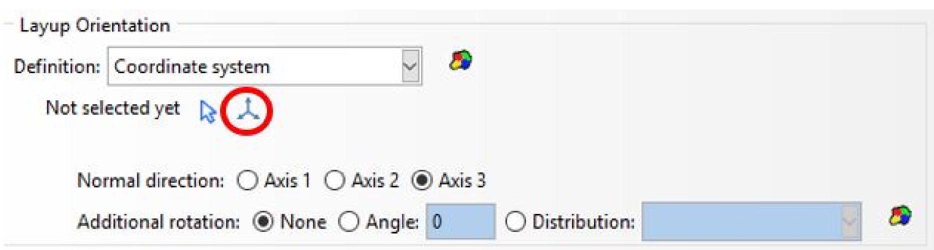


Figure A.15: Specify datum coordinate system for layup

This will open the **Create Datum CSYS** dialog box. Name the datum coordinate system, e.g., ***Datum***, and choose ***Rectangular*** to create a Cartesian coordinate system as shown in Fig. A.16.

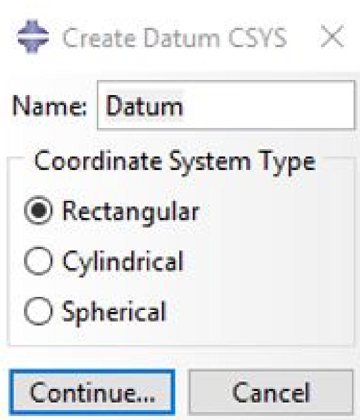


Figure A.16: Create datum coordinate system dialog box

In the mid-surface geometry, choose the bottom left corner (point number 1) in Fig. A.17 as the origin. Next, for the point on the x axis, choose point number 2. Finally, for the point in the $x - y$ plane, pick point number 3. This will create the datum coordinate system. Click **Cancel** to exit the Create Datum CSYS dialog.

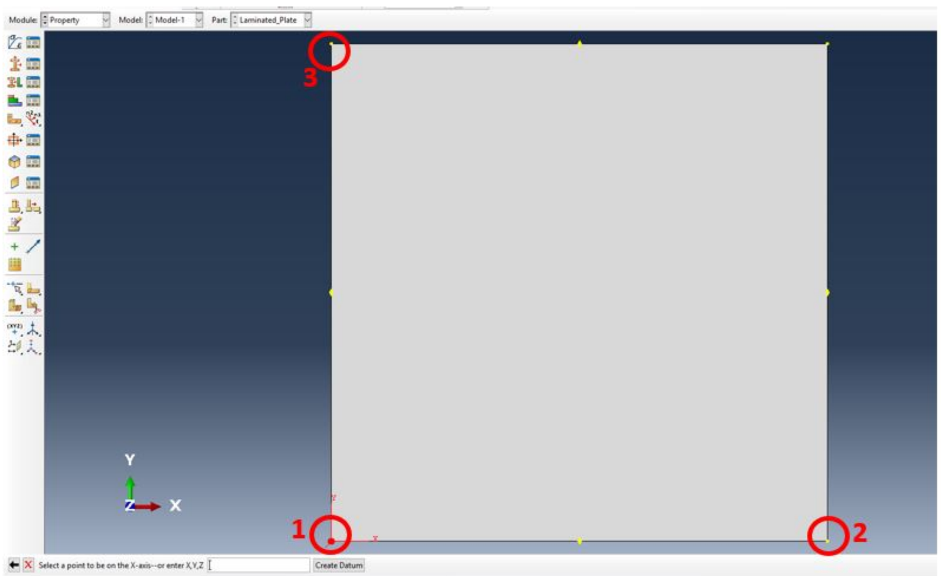


Figure A.17: Module "Property": Layup creation datum creation

Once the datum coordinate system has been created, Abaqus will return to the **Edit Composite Layup** dialog as shown in Fig. A.18. Next, click on  next to **Datum** and select the datum coordinate system you just created as the reference coordinate system for specifying the ply orientations.

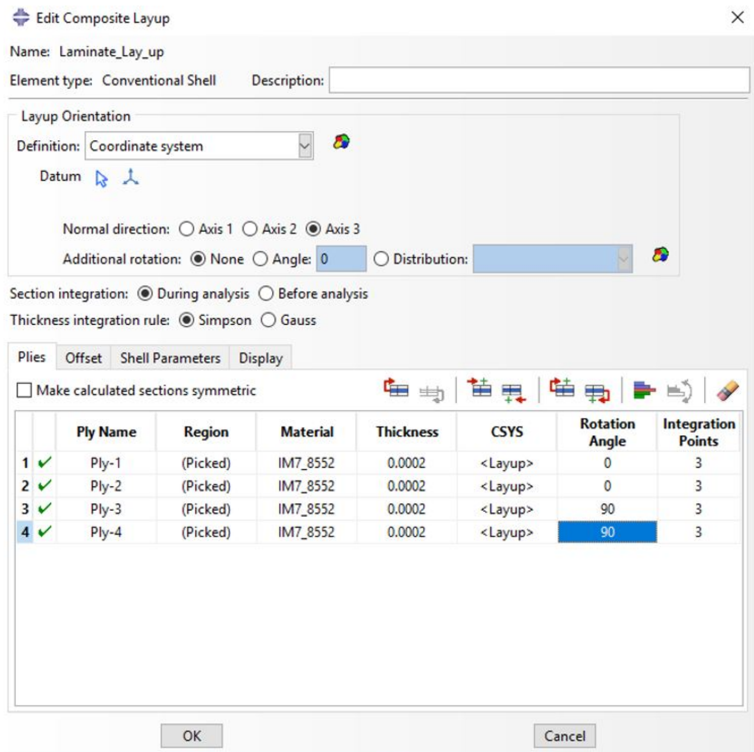


Figure A.18: Specifying the composite layup

Specify the ply layup as follows,

- Choose the **Region** column heading and select the enter part.
- Choose the **Material** column heading and select ***IM7_8552*** as shown in Fig. A.19.
- Choose the **Thickness** column heading and enter **2E-3** (i.e., 2 mm) for ply thickness
- Under **Rotation angle** enter the orientation of each ply relative to the datum coordinate system as shown in Fig. A.18. You will notice that as orientation of each ply is specified, the principal material coordinate system of the ply will be displayed in the Viewport relative to the global datum coordinate system selected as shown in Fig. A.20.

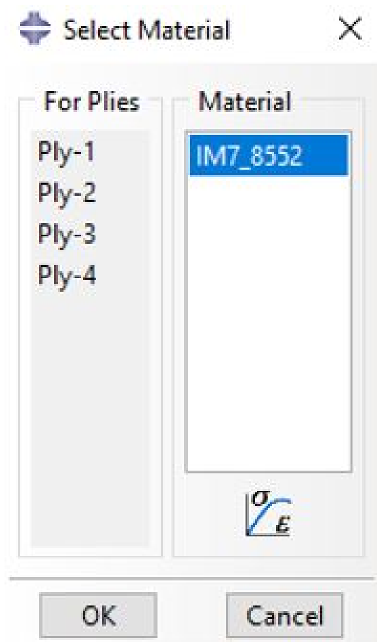
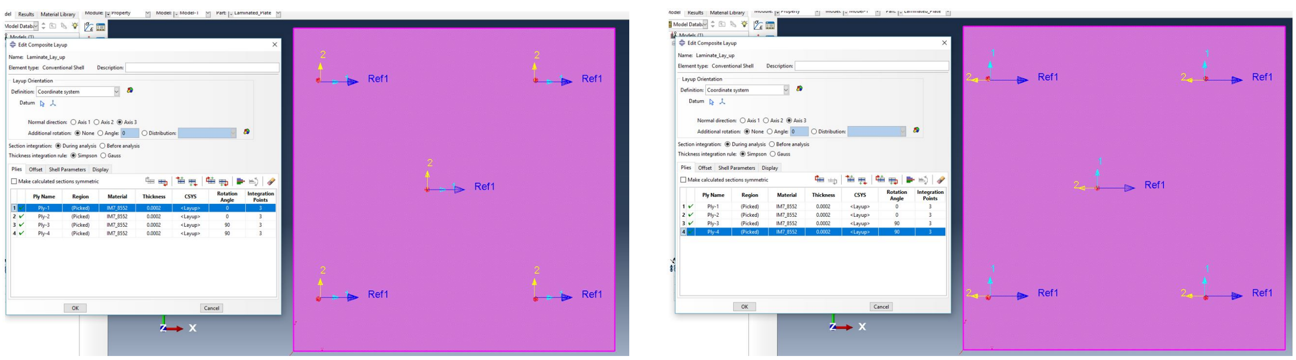


Figure A.19: Select ply material



(a) 0° ply **(b)** 90° ply

Figure A.20: Orientation of principal material coordinate system relative to the datum reference coordinate system

Click **OK** to exit the **Edit Composite Layup** dialog after you have specified the ply orientations.

You can view the full stacking sequence that you just created to verify the ply orientations by clicking on the icon in the top toolbar of your screen. A **Query** window will appear as shown in Fig. A.21. Click on **Ply stack plot** and the select the part in the Viewport as the region for which the composite layup is to be shown. This will display the stacking sequence as shown in Fig. A.21.

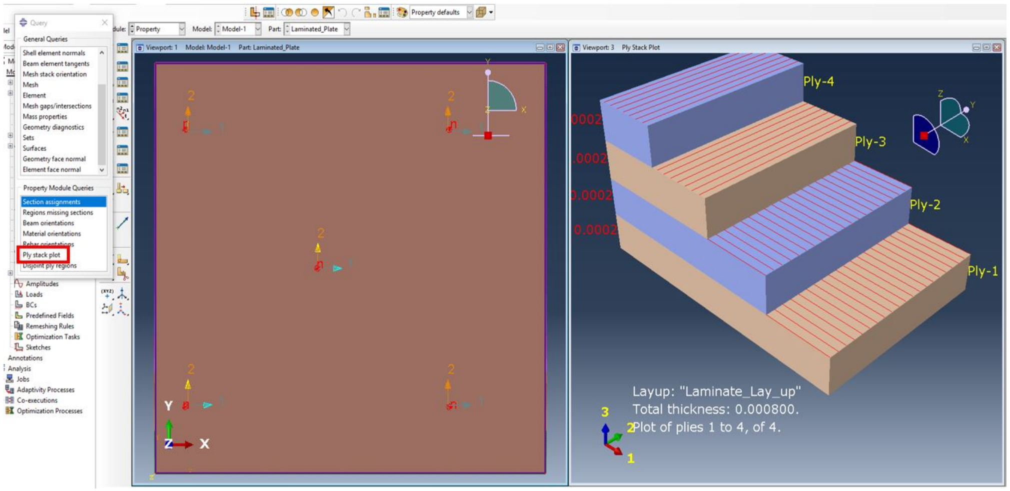
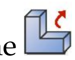


Figure A.21: Stacking sequence visualization

A.5 Module: Assembly

Switch to the **Assembly** module, to create one instance of the plate you created. Click first on the  icon. Then, fill the **Create Instance** dialog box that shows, as shown in Fig. A.22, as follows,

- Under the **Create instance from:** option, choose ***Parts***
- Under **Parts**, pick the part you created, e.g., ***Laminated_Plate***
- Under **Instance Type**, choose ***Independent (mesh or instance)***

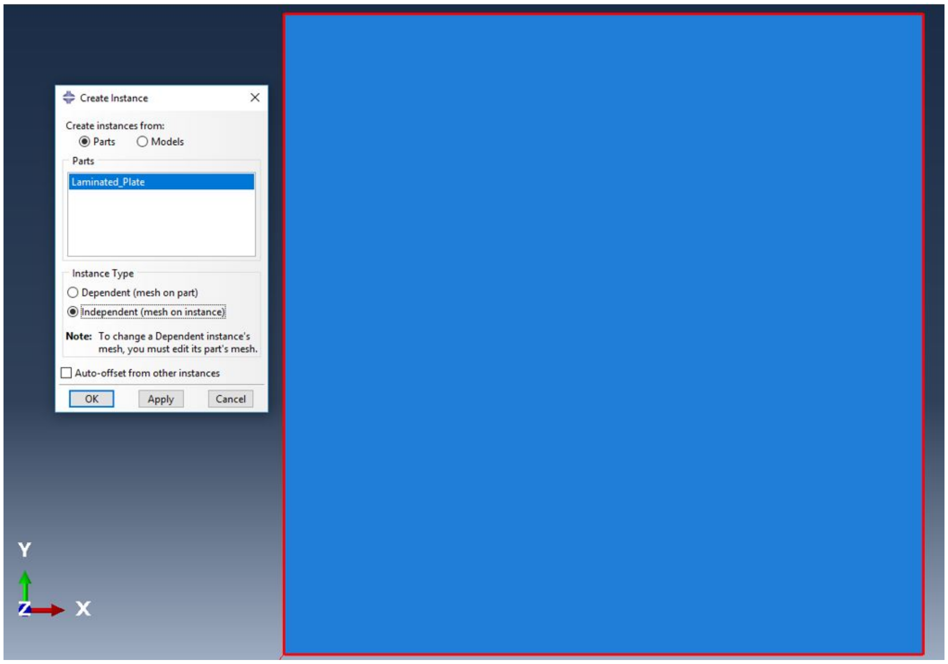


Figure A.22: Assembly creation

Click **OK** to finish creating the instance.

A.6 Module: Step

Switch to the **Step** module to create an analysis step. In this module, click on the  icon. The **Create Step** dialog box, shown in Fig. A.23, will appear

- Enter a name for the analysis step, e.g., ***Static_Analysis***
- Under **Procedure type: General**, choose ***Static, General*** for a static analysis run.

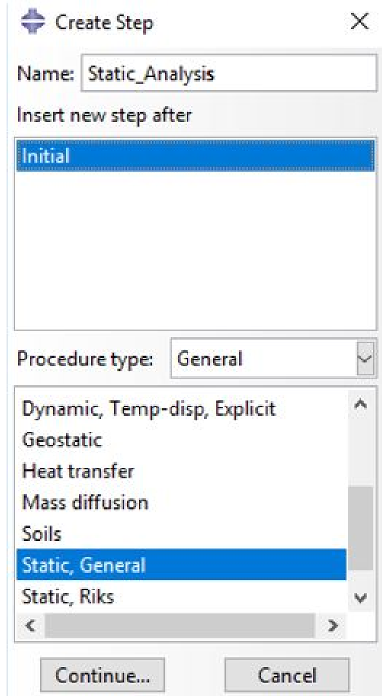
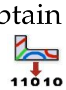


Figure A.23: Create Step dialog box

When you click **OK**, the **Edit Step** dialog box will open. Click **OK** to proceed with the default values.

Abaqus/CAE writes the results of the analysis to the output database (.odb) file. A list of preselected variables are written by default to the output database. In the case of composite materials, we need to specify addition variables to obtain the through-thickness variation of stresses and failure index.

Click on the **Create Field Output**  icon to open the **Create Field** dialog box shown in Fig. A.24. Complete the dialog box as follows,

- Give a name to the additional field outputs, e.g., ***Composite_Field_Outputs***
- Under **Step**, select the step you created, e.g., ***Static_Analysis***
- Click **Continue...**.

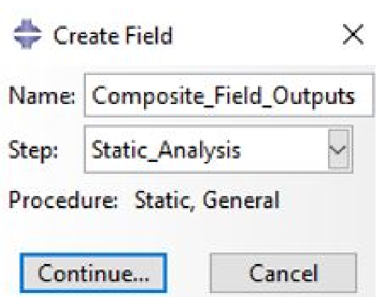


Figure A.24: Create Field dialog box

The **Edit Field Output Request** dialog box will then appear as shown in Fig. A.25. Select the following options to choose the required field outputs for the laminate analysis,

- Under **Domain**, choose ***Composite layup***
- In the **Output Variables** block, select ***Stresses***, ***Strains*** and ***Failure/Fracture*** to get the corresponding outputs through the thickness.
- In the **Output at Section Points** block, select ***All section points in all plies***

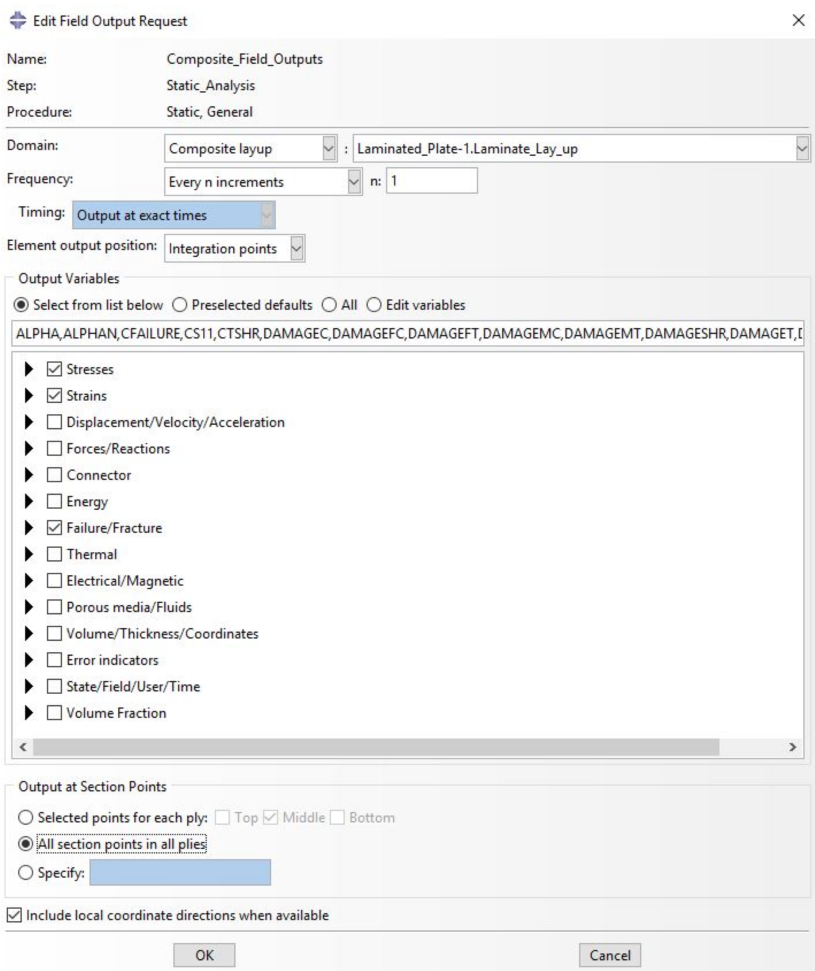


Figure A.25: Create new field output request

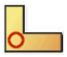
Click **OK** to exit the field output request dialog box.

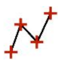
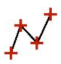
A.7 Module: Load

Switch to the **Load** module to prescribe the loads and boundary conditions.

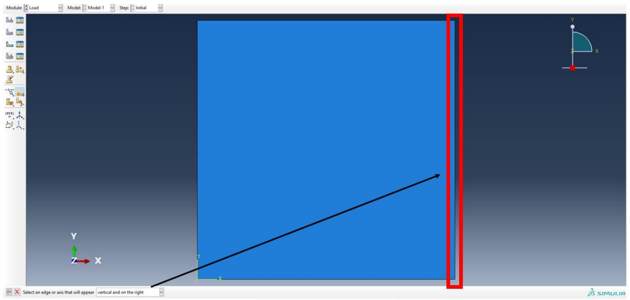
A.7.1 Loads

In this problem a point load is applied. In order to create a point load, we need to first define a point in the geometric domain before we can apply a concentrated load at that point. This is accomplished by partitioning the geometry. Note that it is better to first partition the geometry before specifying the boundary conditions since partitioning the plate will segment the boundaries.

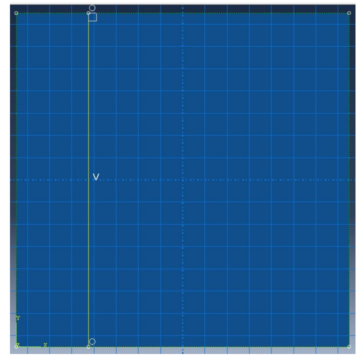
Click on the  icon in the toolbox area to partition a face. The process to partition the plate is as follows,

- Select the right vertical edge of the plate to be the vertical right edge in the Sketcher when partition the face as shown in Fig. A.26(a)
- Click on the  icon and draw a vertical line as shown in Fig. A.26(b). Click on the **Esc** key on your keyboard after drawing the line. Notice the "V" letter in Fig. A.26(b) stands for ***Vertical***
- Click on the  icon to draw the horizontal line of the partition as shown in Fig. A.26(c). Click on the **Esc** key on your keyboard after drawing the line. Notice the "H" letter in Fig. A.26(c) stands for ***Horizontal***
- Dimension your partition by clicking on the  icon. Note that the point of intersection of the two lines will serve as the point of application of the concentrated load. Click on the line and the edge of the plate you are dimensioning with respect to, enter the dimension and press **Enter** on your keyboard after each dimension is entered. Click on the **Esc** key on your keyboard after entering both dimensions. The result is shown in Fig. A.26(d). Note that the dimension "0.07" is actually "0.075" which corresponds to $a/4$ but Abaqus has limited default decimal precision output
- Click on **Done** on the left of the prompt bar to finish the partition.

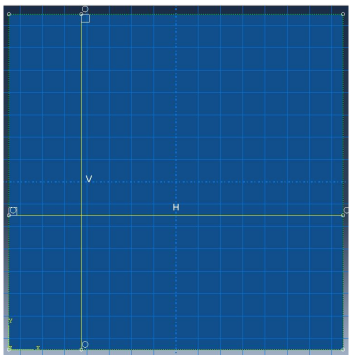
The final partitioned plate is shown in Fig. A.26(e).



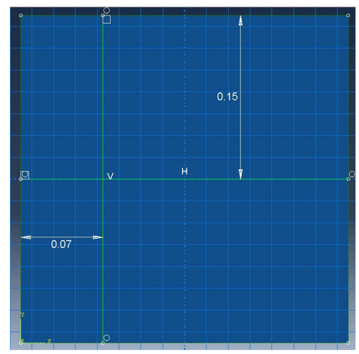
(a) Choosing the right view edge to partition



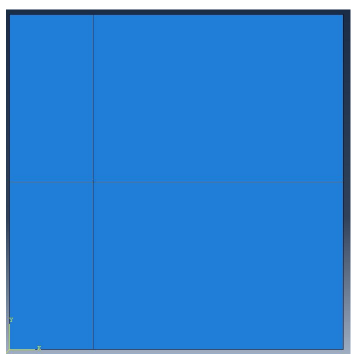
(b) Drawing the first line in the partition of the plate



(c) Drawing the second line in the partition of the plate

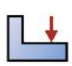


(d) Dimensioning the partition



(e) Final view of the partitioned plate

Figure A.26: Plate partitoning process

To create the point load, click on the  and fill the **Create Load** dialog box as shown in Fig. A.27,

- Give a name to the load to be applied, e.g., ***Point_Load***
- Under **Category**, choose ***Mechanical***
- Under **Types for Selected Step**, choose ***Concentrated force***
- Click **Continue...** on the bottom left of the dialog window

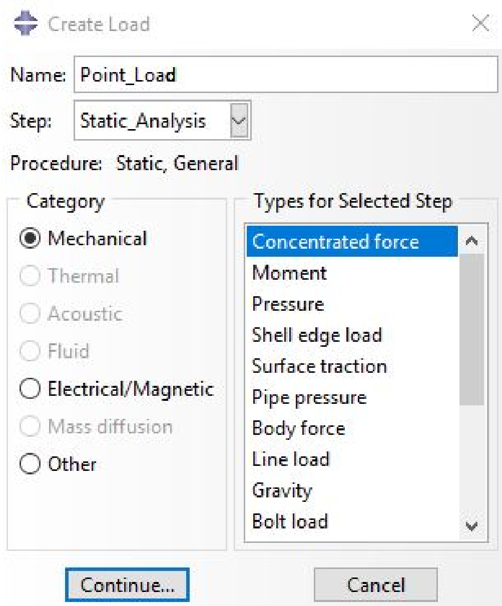


Figure A.27: Create Load dialog box

In the Viewport, click on the point of intersection of all the partitions of the plate, where the load is to be applied, as shown in Fig. A.28. Next, click on **Done** on the left of the prompt bar.

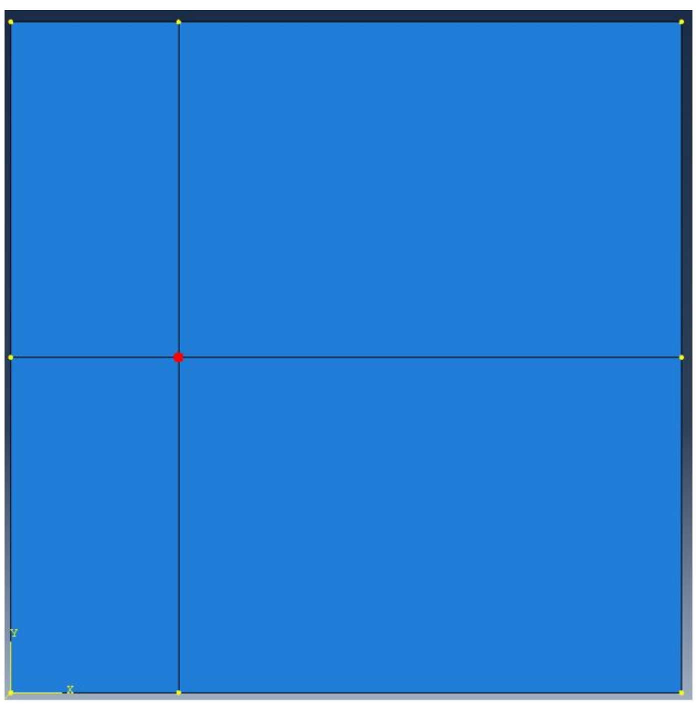


Figure A.28: Selecting the point of application of the concentrated load

The **Edit Load** dialog box will appear as shown in Fig. A.29. Complete the dialog box as follows,

- Under **CF1** and **CF2** enter respectively the x and y components of the load which are both 0
- Under **CF3** enter the z component of the load which is -1 (Newtons)

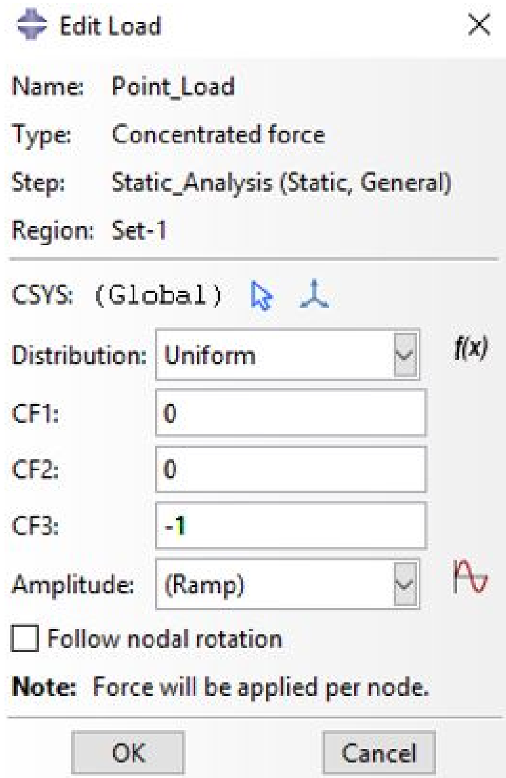


Figure A.29: Concentrated load properties

After finishing the load creation, if you rotate the model using the rotation tool  in the menu bar, you can view the concentrated force as shown in Fig. A.30.

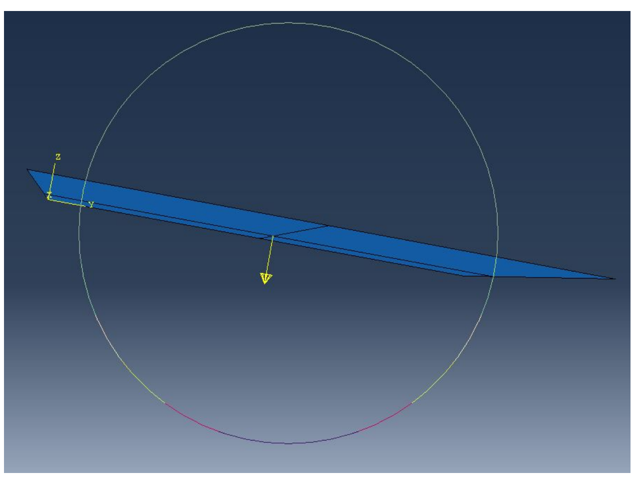
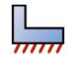


Figure A.30: Module "Load": Final view

A.7.2 Boundary conditions

To create the S_2 boundary conditions on the edges, click on the  icon. Complete the **Create Boundary Conditions** dialog box, as shown in Fig. A.31,

- Give a name to the boundary condition, e.g., ***S2_BC_x_0a***
- Under **Category**, choose ***Mechanical***
- Under **Types for Selected Step**, choose ***Displacement/Rotation*** to restrain the appropriate mid-surface displacements and rotations on the simply supported edges

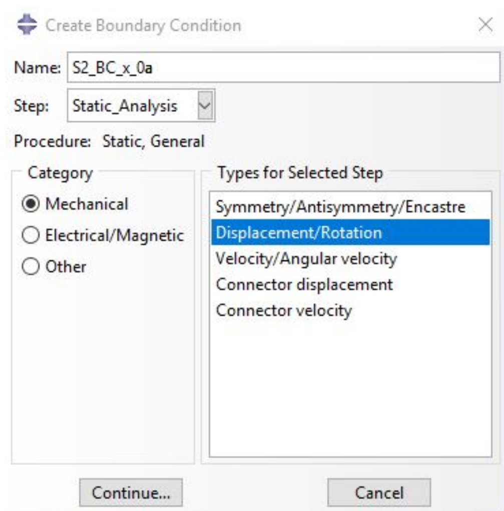


Figure A.31: S_2 boundary conditions creation on the edges $x = 0, a$ dialog box

Pick the edges where this boundary condition will be applied as shown in Fig. A.32. Note to select multiple edges, press and hold the **Shift** key on your keyboard to select all the edges.

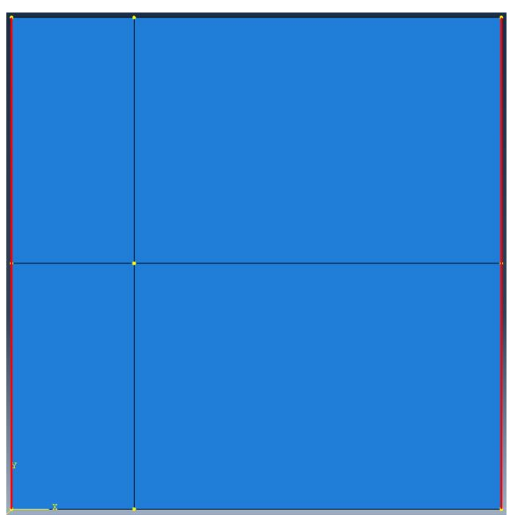


Figure A.32: $x = 0, a$ sides selection

Then, click on **Done** on the left of the prompt bar. The **Edit Boundary Conditions** dialog box will appear as shown in Fig. A.33. Fill the dialog box as follows,

- Select **U2** and **U3** which represent respectively v_0 and w_0
- Set the values to **0** since they are restrained in S_2 boundary conditions
- Click on **OK** on the bottom left of the dialog box.

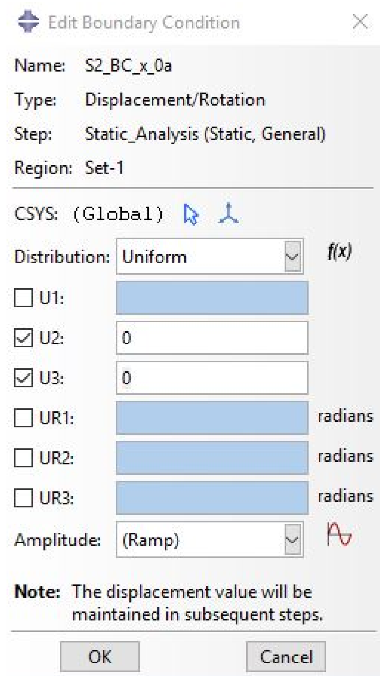


Figure A.33: S_2 boundary conditions at $x = 0, a$

Repeat the process for the edges $y = 0$ and $y = b$, where $U_3 = 0$ and $U_1 = 0$ (i.e., $u_0 = 0$) instead of $U_2 = 0$.

A.8 Module: Mesh

Switch to the **Mesh** module. We note that the Abaqus student edition limits the simulation to a maximum of 1000 nodes. Therefore, the number of elements is chosen accordingly. In practical applications, it is important to perform a convergence analysis by increasing the number of elements. This can be done using the research edition of Abaqus which does not restrict the number of nodes. Since the purpose of this tutorial is to provide an overview of the modeling process, we will perform the analysis using a fixed number of nodes.

First, you should choose the element type. Click on the **Assign Element Type** 

icon. Next, select the entire plate when prompted for the region to be assigned an element type and click **Done**. This will open the **Element Type** dialog box as shown in Fig. A.34 will appear. We will use the default **S4R element** (4-node doubly curved thin or thick shell with reduced integration). Click **OK** and **Done**.

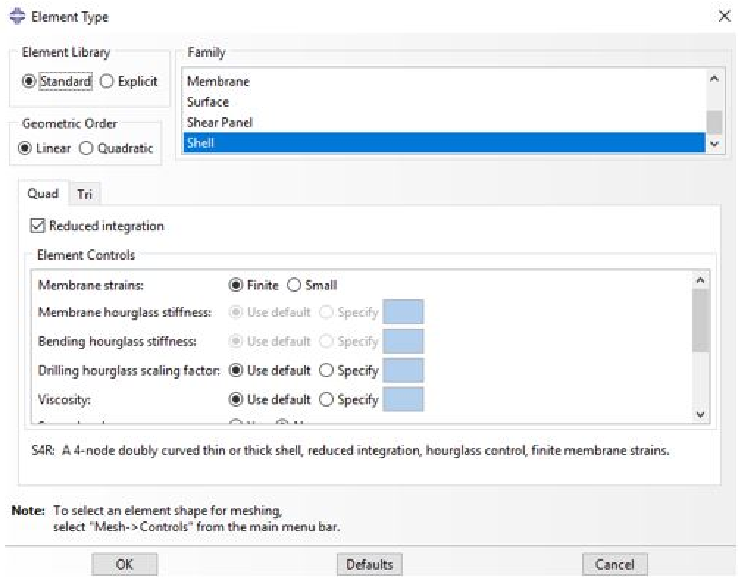
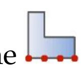


Figure A.34: Module "Mesh": Element type

You will need to seed the edges of the model to create a mesh. Click on the  icon to starting seeding the edges. Then, select the four edge segments shown in bold red line in Fig. A.35. You can press and hold the **Shift** key to select multiple edges. Note that different numbers of seeding points will be used for the other edge segments which are of different length.

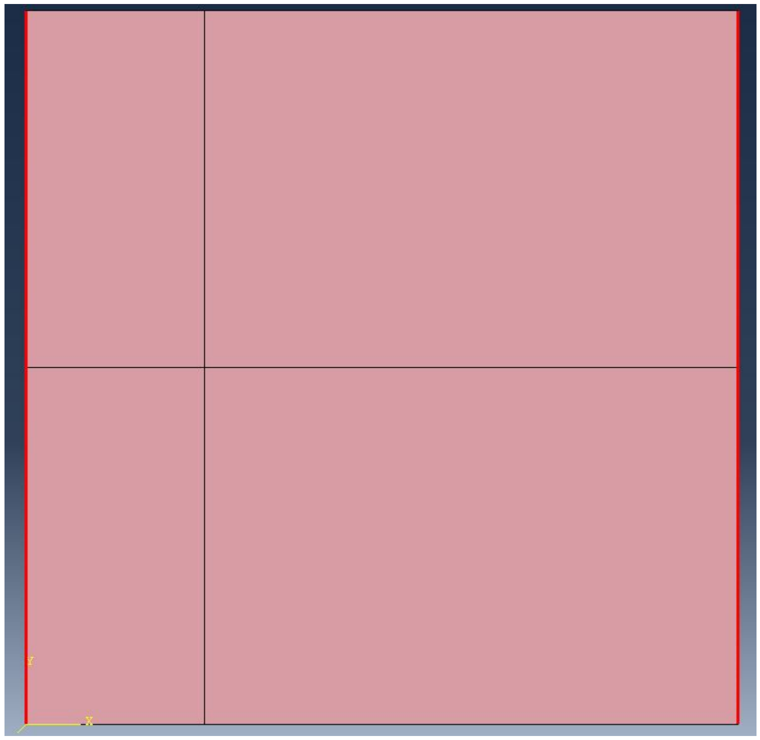


Figure A.35: Selecting the edges to seed

After selecting the four edge segments to be seeded, click **Done** on the left of the prompt bar. The **Local Seeds** dialog box shown in Fig. A.36 will then appear. Fill the dialog box as follows,

- Under **Method**, choose ***By number*** to seed with numbers of elements instead of sizes of elements
- For **Number of elements**, we recommend entering 15 to get 15 elements along each of the chosen edges
- Click **OK** on the bottom left of the dialog box.

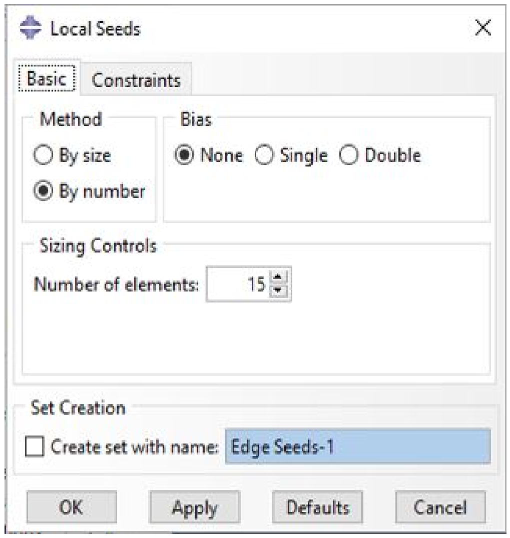
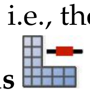


Figure A.36: Edge seeding dialog box

This process is repeated for the other four edge segments corresponding to $y = 0$ and $y = b$. We recommend 7 elements for the shorter edge segments that extend from $x = 0$ to $a/4$ and 21 elements for the longer edges from $x = a/4$ to a. Click **Done** after seeding all the edges.

Next, to make sure the mesh is structured, i.e., the mid-point of the plate corresponds to a node for example, click on the **Assign Mesh Controls**  icon. Then, select the whole plate, i.e., all partitions, and press **Enter** on your keyboard. The dialog box shown in Fig. A.37 will appear. Fill the dialog box as follows,

- Under **Element Shape**, choose ***Quad*** so that your mesh is exclusively quadrilateral elements
- Under **Technique**, choose ***Structured*** to obtain a structured mesh with elements arranged in a regular grid

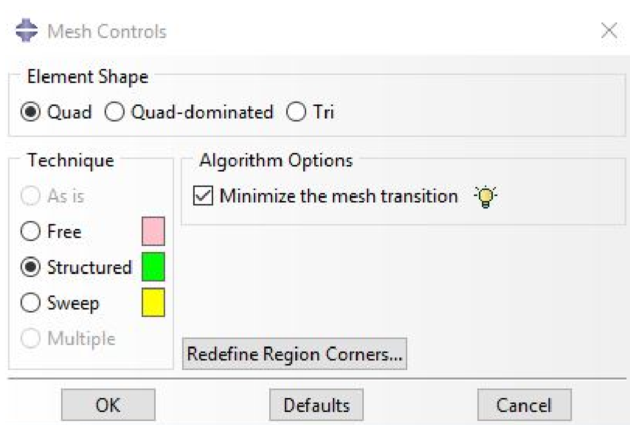


Figure A.37: Finite element mesh controls

Finally, click on the **Mesh Part Instance** icon in the toolbox area and click **OK** when prompted to mesh the part instance. The instance is then meshed, and the result should look similar to the mesh shown in Fig. A.38 if the same number of edge seeds were used.

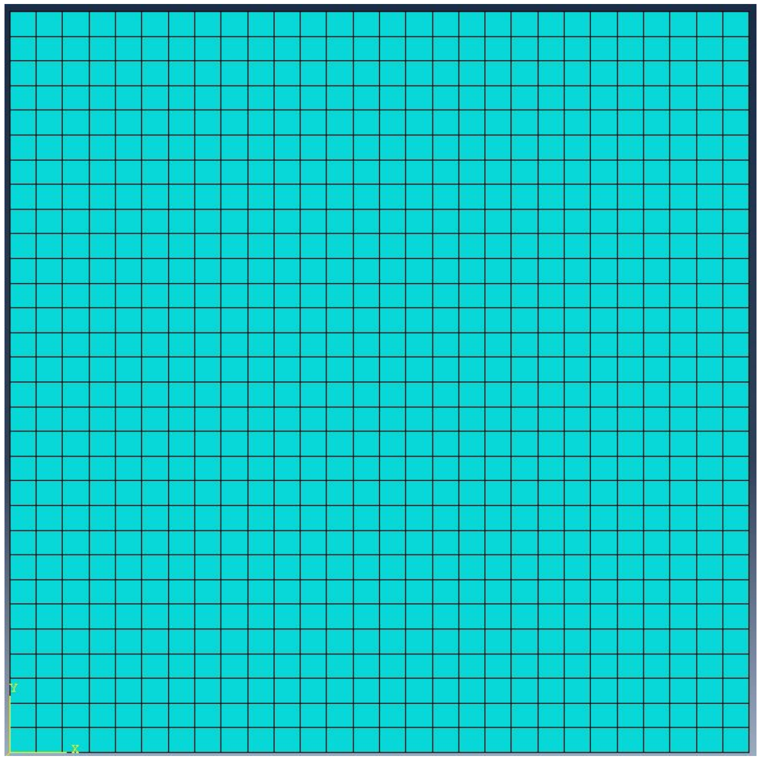


Figure A.38: Finite element mesh consisting of four-noded elements

A.9 Module: Job

Switch to the ***Job*** module to create an analysis job to perform the simulation. Click on the icon in the toolbar area to open the **Create Job** dialog box. Enter a name for the job, e.g., ***Laminate_Job***, as shown in Fig. A.39 and click **Continue...** on the bottom left of the dialog box. This will open the **Edit Job** dialog box. Click **OK** to proceed with the default options.

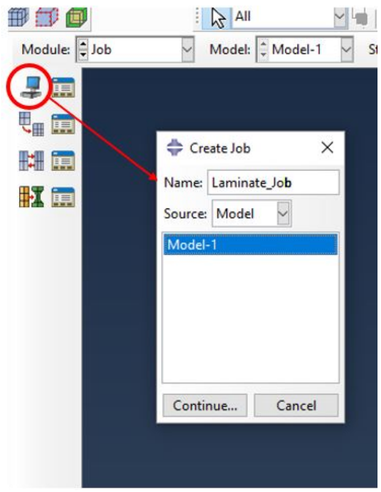


Figure A.39: Job creation

To submit the job, click on the icon in the toolbar area to open the **Job Manager** dialog box shown in Fig. A.40. Choose the job and click on **Submit** to perform the finite element analysis.

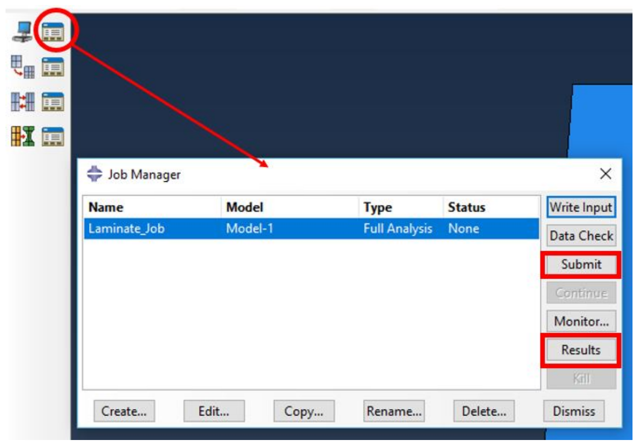


Figure A.40: Job submission

After the run **Status** shows **Completed**, click on **Results** to view the results.

A.10 Module: Visualization

Now that we have performed the analysis, we can postprocess the data and visualize the results. The meshed plate, shown in Fig. A.41, will appear at first.

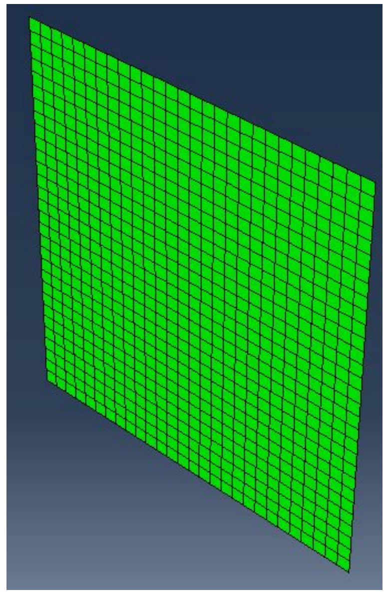
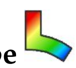


Figure A.41: Visualization module initial view

A.10.1 Displacements

In order to plot the deformed configuration, click on the **Plot Contours on Deformed Shape**  icon. Then, to specifically view the vertical displacement, i.e., U3 in Abaqus, go to the context bar, choose **U** and **U3** to visualize the vertical displacements, shown in Fig. A.42. It can be seen that the maximum deflection is -6.637×10^{-4} m. The deformed shape of the plate is shown as well.

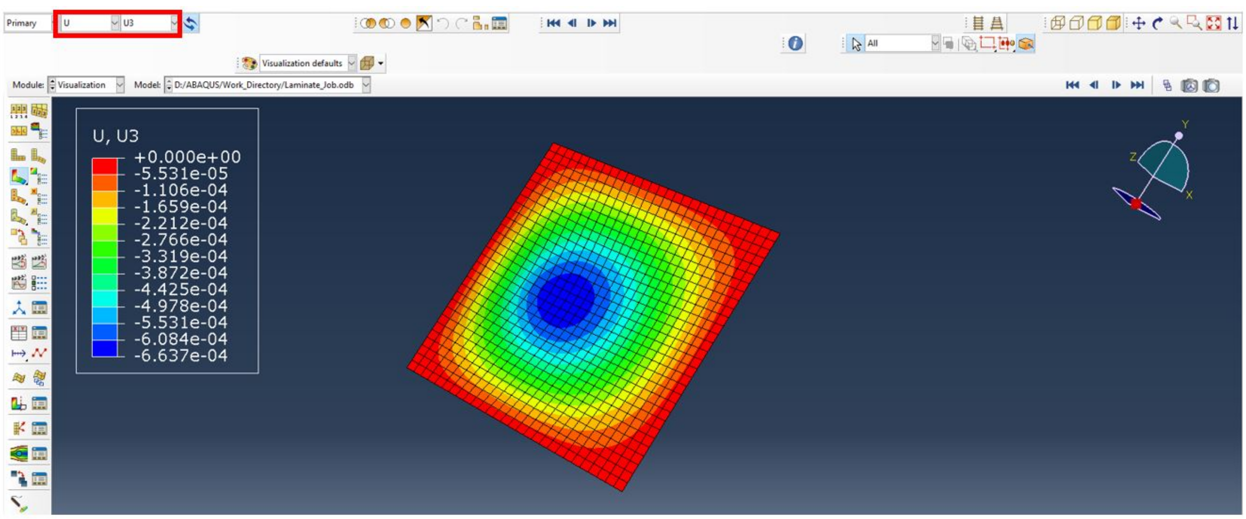


Figure A.42: Module "Visualization": Deflection U_3

A.10.2 Probing for values

To be able to extract the displacement value at a specific point, you need to probe the value. If the vertical displacement of the middle of the plate is to be determined for example, go to the menu bar, under **Tools → Query**, the dialog box shown in Fig. A.43 appears.

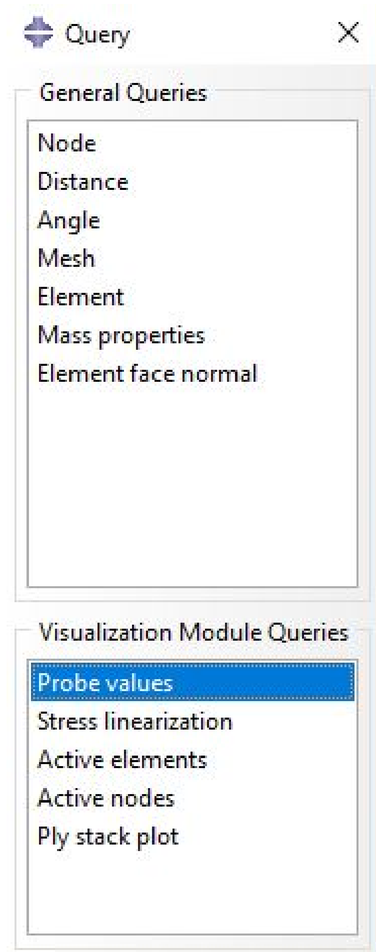


Figure A.43: Query dialog box

In this dialog box, click on **Probe values** to be able to pick nodes with your cursor on the plate geometry and get their properties. After clicking on **Probe values**, another dialog box, shown in Fig. A.44, will appear. Under **Probe**, choose ***Nodes*** to probe nodal values.

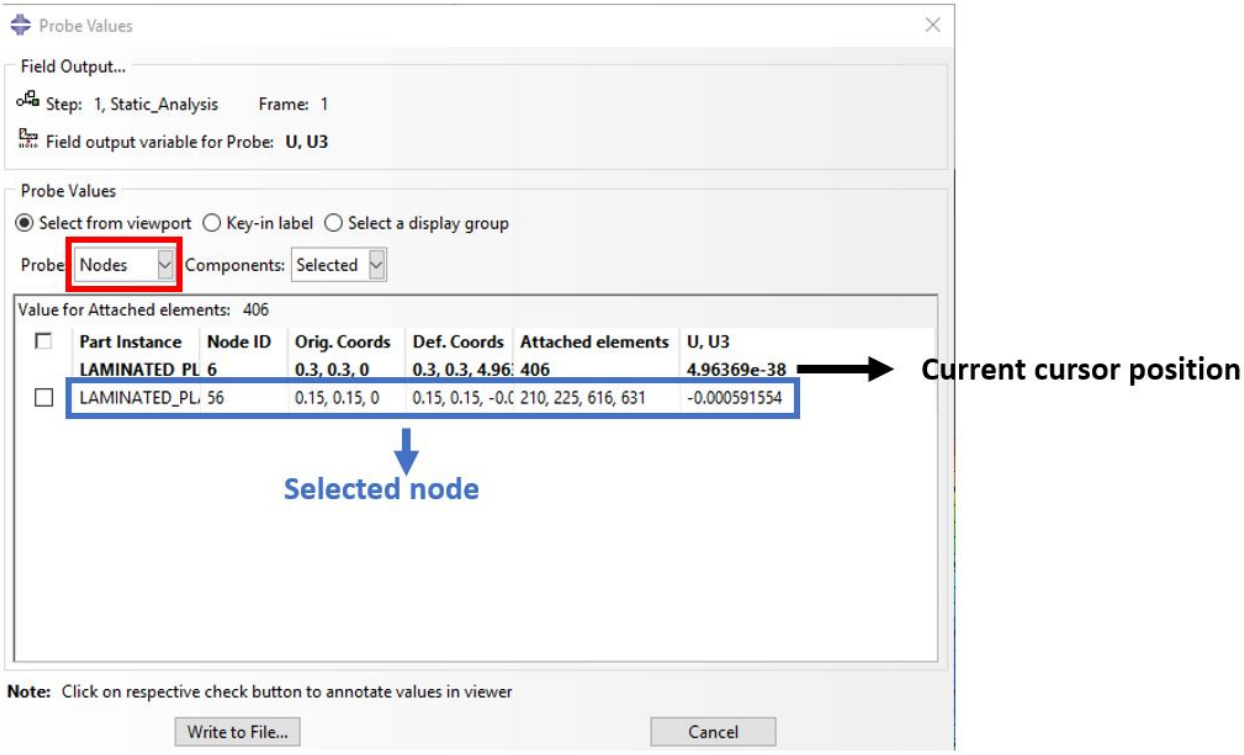


Figure A.44: Probing for values dialog box

As you move your cursor on the plate, the node properties of the node corresponding to your current cursor position will show in the **Probe Values** dialog box. If you click on a node, i.e., select a node, its properties will be fixed as shown in Fig. A.44. Each column corresponds to a property of the node as follows,

- **Node ID** represents the node number in the mesh created by Abaqus
- **Orig. Coords** represents the undeformed, i.e., original, coordinates of the node
- **Def. Coords** represents the coordinates of the node after deformation
- **Attached elements** represents the element numbers, i.e., ID's, of the elements connected to this node
- **U, U3** represents the vertical displacement of the node

Note that **U, U3** was a part of the probe values because it was selected earlier in the context bar as shown in Fig. A.42. The variable you choose in the context bar will appear as a column in the **Probe Values** dialog box. It can be seen in Fig. A.44 that the vertical displacement at the middle of the plate is of −0.000591554 m or 0.000591554 m downwards.

A.10.3 Stresses

To plot the stresses, go to the context bar and change **U** and **U3** to **S** and **S11**. Note that **S11** corresponds to σ_1 in the material coordinate system and **S22** corresponds to σ_2 in the material coordinate system. The stress depends on the through-thickness location of a point, i.e., the z coordinate. To choose where you want the stress plot, go to the menu bar under **Results → Section Points**. The dialog box shown in Fig. A.45 appears. Under **Selection method** choose ***Plies*** to pick stress plotting locations based on the plies.

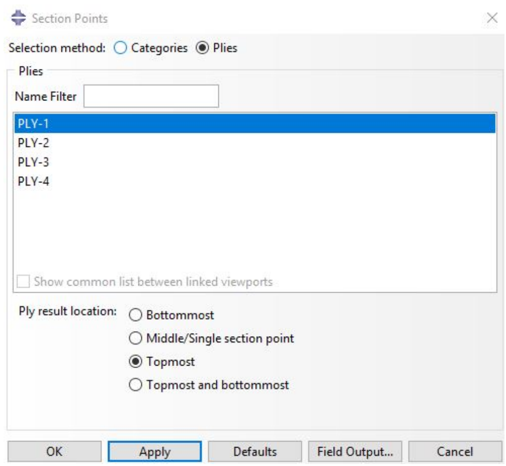


Figure A.45: Dialog box to choose the stress location to plot

To plot the stress at the top of the first ply, select the following in the dialog box shown in Fig. A.45,

- Choose ***PLY-1***
- Under **Ply result location**, choose ***Topmost***

The result is shown in Fig. A.46

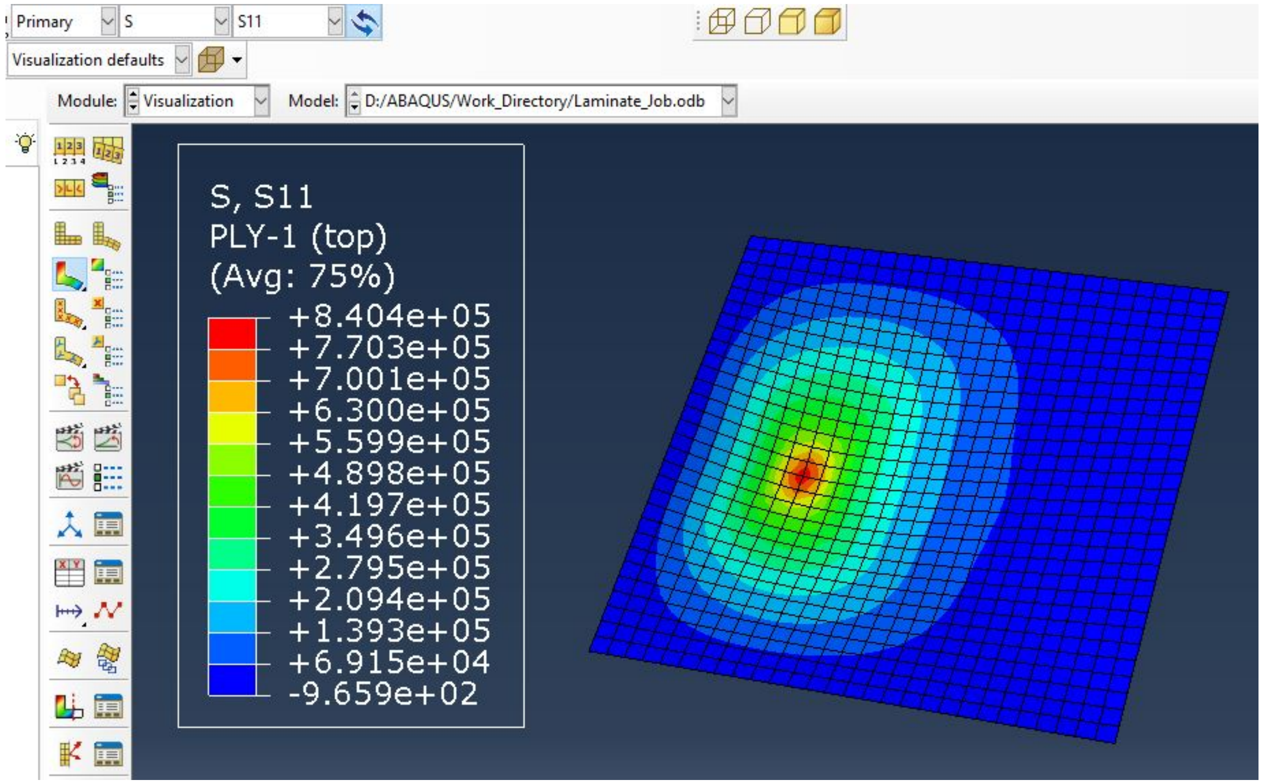


Figure A.46: Stress S11 variation on the top surface of the first ply in the laminate

Note if you choose under **Ply result location** the option **Topmost and bottommost**, the result will show a plate where the contour on the top surface corresponds to the distribution of **S11** on the top surface of the corresponding ply and the bottom side contour corresponds to the stress distribution on the bottom surface of the ply.

A.10.4 Through-thickness plots

To plot the through thickness variation of a parameter at a certain point, go to the menu bar, then go under **Tools → xy Data → Create...**. The **Create XY Data** dialog box shown on the left of Fig. A.47, will appear.

- Under **Source**, choose ***Thickness*** which means you want the through thickness plots
- Click on **Continue...** at the bottom left of the dialog box

The **XY Data From Shell Thickness** dialog box will appear. Under the tab **Variables**, check the following,

- Under **Position**, choose **Element Nodal**, to pick a location by specifying a corresponding element ID of an element connected to the node at this location

- Under **Click checkboxes or edit the identifiers shown next to Edit below.**, check **S11** for σ_1, and later **TSAIW: Tsai-Wu failure measure** for the Tsai-Wu failure index

Since the middle of the plate is defined by a node of ID 56, Fig. A.44, and one of the elements connected to it has an ID of 210, under the **Elements** tab of the **XY Data From Shell Thickness** dialog box shown in Fig. A.47,

- Go to the **Elements** portion
- Under **Method**, choose ***Element labels*** to choose the location by the label of an element connected to the node at the location
- Under **Element labels**, enter the element ID of an element connected to the node located at the middle of the plate, e.g., 210
- Click on **Plot** at the middle of the bottom of the dialog box

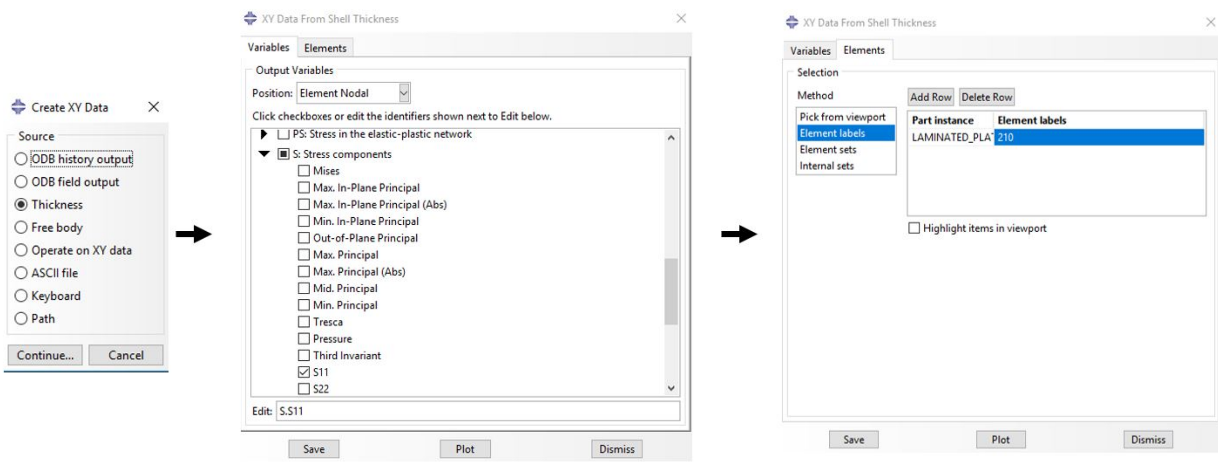


Figure A.47: Steps to plot the through thickness variation of S11

Figure A.48 shows the result of the plot. As it can be seen there are four superimposed plots each corresponding to a node of element 210. To see only the curve corresponding to the mid-point, we need to delete the other three curves.

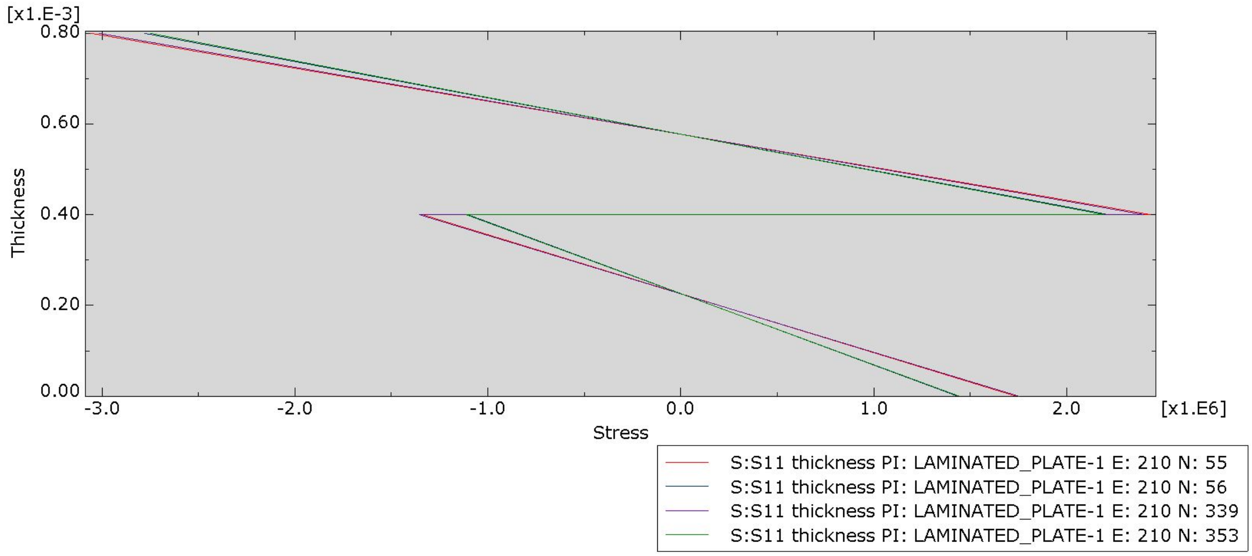


Figure A.48: S11 plots corresponding to the four nodes attached to element 210

To delete the curves that are not needed, go to the menu bar under **Tools → xy Data → Manage**, then the window shown in Fig. A.49 will appear.

- Select, as shown, the XY data not corresponding to the mid-point, i.e., curves for the other three nodes in element 210 that are not node 56
- Click on **Delete...**
- In the dialog box that shows up, click on **Yes**
- Close **XY Data Manager** dialog box

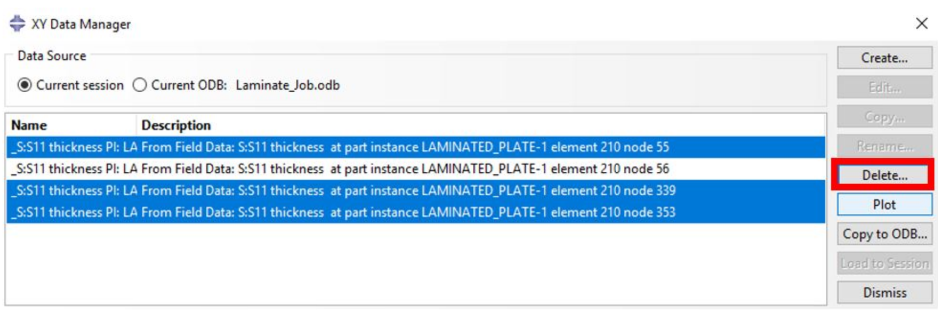


Figure A.49: Deleting the plots at the nodes connected to element 210, other than node 56

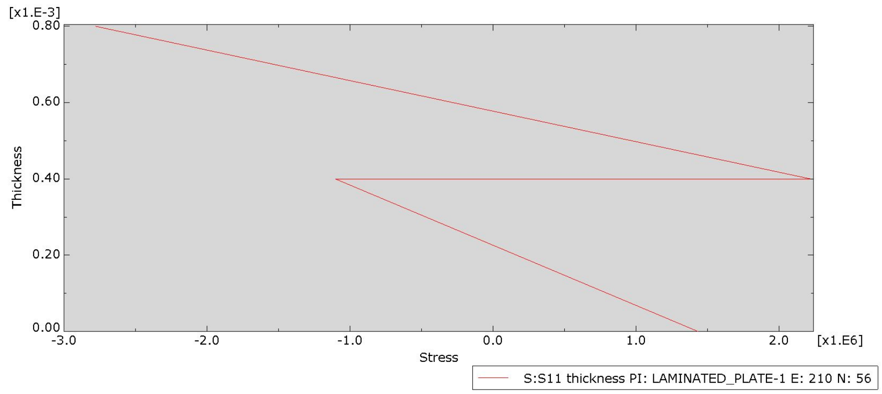


Figure A.50: Through-thickness variation of the stress component S11 at the center of the plate

In Abaqus, the failure index R is defined as the scaling factor such that, for a given stress state $\{\sigma_1, \sigma_2, \tau_{12}\}$,

$$g\left(\frac{\sigma_1}{R}, \frac{\sigma_2}{R}, \frac{\tau_{12}}{R}\right) = 1 \tag{A.2}$$

where $g(\sigma_1, \sigma_2, \tau_{12})$ is defined in (2.41). Note that the scaling factor $1/R$ multiplies all of the stress components simultaneously such that the resulting stress state lies on the failure envelope. If the failure index $R < 1$, then the stress state lies within the failure envelope. Values of $R \geq 1$ indicate failure.

By definition, the failure index R is the reciprocal of the safety factor S_{fa}, i.e,

$$R = \frac{1}{S_{fa}} \tag{A.3}$$

The through-thickness variation of the Tsai-Wu failure index R can be plotted using a process similar to that for the stresses as shown in Fig. A.51

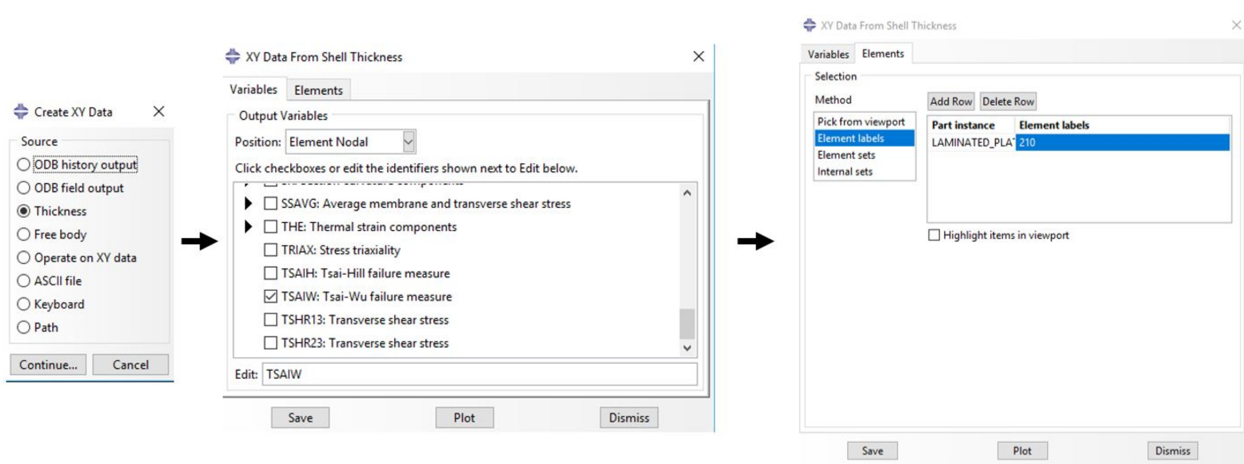


Figure A.51: Steps to plot the through thickness variation of the Tsai-Wu failure index

After deleting the additional curves at the other nodes of element 210, we obtain the through-thickness variation shown in Fig. A.52.

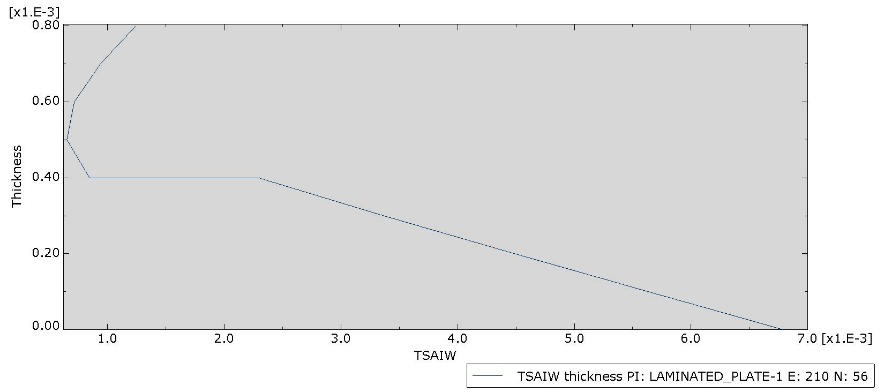


Figure A.52: Through-thickness variation of the Tsai-Wu failure index at the middle of the laminated plate

The Tsai-Wu theory predicts that the laminated plate will not fail since $R < 1$ throughout.

B

Matlab Code for Laminated Composite Structures

B.1 Material Properties

Unidirectional carbon fiber-reinforced composite

```
function [E1,nu12,E2,G12,F1t,F1c,F2t,F2c,F6,h,rho] = UnidirectionalCarbonEpoxyProperties
% Assigns material properties in the principal material directions
%
%   Syntax:
%     E1,nu12,E2,G12,F1t,F1c,F2t,F2c,F6,h,rho] = UnidirectionalCarbonEpoxyProperties
%
%   Inputs: None
%
%   Output:
%     E1 - Young's modulus in the 1-direction
%     nu12 - major Poisson's ratio
%     E2 - Young's modulus in the 2-direction
%     G12 - inplane shear modulus
%     F1t - The tensile strength in the 1-material direction
%     F1c - The compressive strength in the 1-material direction
%     F2t - The tensile strength in the 2-material direction
%     F2c - The compressive strength in the 2-material direction
%     F6  - The shear strength in the 1-2 material plane
%     Note 1 and 2 are the principal material directions.
%     Typically E1,E2,G12,F1t,F2t,F2t, F2c and F6 are specified in SI units of Pa.
%
%   Author: Senthil S. Vel, University of Maine
%
%   See also ReducedStiffness, ReducedCompliance, OffAxisStiffness, OffAxisCompliance.

% Representative elastic properties of an IM7/8552 unidirecional
% fiber-reinforced carbon fiber composite

% The Young's moduli and shear moduli are in Pa
```

```
E1 = 167.4e9;
nu12 = 0.30;
E2 = 10.3e9;
G12 = 6.4e9;

% Representative strengths of a lamina in Pa
F1t = 2723e6;
F1c = 1689e6;
F2t = 64.1e6;
F2c = 199.8e6;
F6 = 92.3e6;

% Density in kg/m^3
rho = 1588;

% Ply thickness in meters
h = 0.2e-3;

% Print lamina properties
fprintf('Elastic moduli of the composite material: \n')
fprintf('  E1 = %g GPa \n',E1/1e9)
fprintf('  nu12 = %g \n',nu12)
fprintf('  E2 = %g GPa \n',E2/1e9)
fprintf('  G12 = %g GPa \n',G12/1e9)

fprintf('Strengths of the composite material: \n')
fprintf('  F1t = %g MPa \n',F1t/1e6)
fprintf('  F1c = %g MPa \n',F1c/1e6)
fprintf('  F2t = %g MPa \n',F2t/1e6)
fprintf('  F2c= %g MPa \n',F2c/1e6)
fprintf('  F6 = %g MPa \n',F6/1e6)

fprintf('Density: \n')
fprintf('  rho = %g kg/m^3 \n',rho)

fprintf('Ply thickness: \n')
fprintf('  h = %g mm \n\n',h/1e-3)
```

B.2 Lamina Functions

Reduced stiffness matrix $[Q]$

```
function Q = ReducedStiffness(E1,nu12,E2,G12)
% ReducedStiffness calculates the plane stress reduced elastic
% stiffness matrix [Q] for a composite lamina.
%
%   Syntax:
%     Q = ReducedStiffness(E1,nu12,E2,G12)
%
%   Inputs:
%     E1 - Young's modulus in the 1-direction
%     nu12 - major Poisson's ratio
%     E2 - Young's modulus in the 2-direction
%     G12 - inplane shear modulus
%     Note 1 and 2 are the principal material directions.
%     Typically E1, E2 and G12 are specified in SI units of Pa.
%
%   Output:
%     Q - 3x3 reduced stiffness matrix for a composite lamina
%
%   Author: Senthil S. Vel, University of Maine
%
%   See also ReducedCompliance, OffAxisStiffness, OffAxisCompliance.

%   Calculate the minor Poisson's ratio using the reciprocal relations
    nu21 = nu12*E2/E1;

%   Evaluate the elements of the reduced stiffness matrix
    Q11 = E1/(1-nu12*nu21);
    Q12 = nu12*E2/(1-nu12*nu21);
    Q22 = E2/(1-nu12*nu21);
    Q66 = G12;

%   Arrange the elements to form the reduced stiffness matrix [Q]
    Q = [Q11  Q12  0;
         Q12  Q22  0;
          0    0  Q66];

end
```

Reduced compliance matrix $[S]$

```
function S = ReducedCompliance(E1,nu12,E2,G12)
% ReducedCompliance calculates the plane stress reduced elastic
% compliance matrix [S] for a composite lamina.
%
%   Syntax:
%     S = ReducedCompliance(E1,nu12,E2,G12)
%
%   Inputs:
%     E1 - Young’s modulus in the 1-direction
%     nu12 - major Poisson’s ratio
%     E2 - Young’s modulus in the 2-direction
%     G12 - inplane shear modulus
%     Note 1 and 2 are the principal material directions.
%     Typically E1, E2 and G12 are specified in SI units of Pa.
%
%   Output:
%     S - 3x3 reduced complaince matrix for a composite lamina.
%
%   Author: Senthil S. Vel, University of Maine
%
%   See also ReducedStiffness, OffAxisStiffness, OffAxisCompliance.

%   Evaluate the elements of the reduced compliance matrix
    S11 = 1/E1;
    S12 = -nu12/E1;
    S22 = 1/E2;
    S66 = 1/G12;

%   Arrange the elements to form the reduced compliance matrix [S]
    S = [S11  S12  0;
         S12  S22  0;
          0    0  S66];

end
```

Off-axis stiffness matrix $[\bar{Q}]$

```
function QBar = OffAxisStiffness(Q,Theta)
% OffAxisStiffness calculates the plane stress reduced elastic
% stiffness matrix [QBar] for an off-axis composite lamina.
%
```

```
%   Syntax:
%     QBar = OffAxisStiffness(Q,Theta)
%
%   Inputs:
%     Q - 3x3 plane-stress reduced stiffness matrix for a composite lamina
%     Theta - Angle in degrees from the x-axis to the 1-axis (CCW positive)
%
%   Output:
%     QBar - 3x3 reduced stiffness matrix for an off-axis lamina
%
%   Author: Senthil S. Vel, University of Maine
%
%   See also ReducedCompliance, ReducedStiffness, OffAxisCompliance.

% Cosine and Sine of the angle
  m = cosd(Theta);
  n = sind(Theta);

% 2D reduced stiffness matrix (Q) values extraction
  Q11 = Q(1,1); Q12 = Q(1,2); Q22 = Q(2,2); Q66 = Q(3,3);

% Calculate the off-axis stiffnesses QBar
  QBar11 = Q11*m^4+2*(Q12+2*Q66)*m^2*n^2+Q22*n^4;
  QBar12 = (Q11+Q22-4*Q66)*m^2*n^2+Q12*(m^4+n^4);
  QBar16 = (Q11-Q12-2*Q66)*n*m^3+(Q12-Q22+2*Q66)*n^3*m;
  QBar22 = Q11*n^4+2*(Q12+2*Q66)*n^2*m^2+Q22*m^4;
  QBar26 = (Q11-Q12-2*Q66)*n^3*m+(Q12-Q22+2*Q66)*n*m^3;
  QBar66 = (Q11+Q22-2*Q12-2*Q66)*n^2*m^2+Q66*(n^4+m^4);

% Assemble the QBar matrix
  QBar =[QBar11 QBar12 QBar16;
         QBar12 QBar22 QBar26;
         QBar16 QBar26 QBar66];

end
```

Off-axis compliance matrix $[\bar{S}]$

```
function SBar = OffAxisCompliance(S,Theta)
% OffAxisCompliance calculates the plane stress reduced elastic
% compliance matrix [SBar] for an off-axis composite lamina.
%
%   Syntax:
%     SBar = OffAxisCompliance(S,Theta)
```

```
%
%   Inputs:
%     S - 3x3 plane-stress reduced compliance matrix for a composite lamina
%     Theta - Angle in degrees from the x-axis to the 1-axis (CCW positive)
%
%   Output:
%     SBar - 3x3 reduced complaince matrix for an off-axis lamina
%
%   Author: Senthil S. Vel, University of Maine
%
%   See also ReducedCompliance, ReducedStiffness, OffAxisStiffness.

% Cosine and Sine of the angle
  m = cosd(Theta);
  n = sind(Theta);

% 2D reduced compliance matrix (S) values extraction
  S11 = S(1,1); S12 = S(1,2); S22 = S(2,2); S66 = S(3,3);

% Calculate the off-axis compliances SBar
  SBar11 = S11*m^4+(2*S12+S66)*m^2*n^2+S22*n^4;
  SBar12 = (S11+S22-S66)*m^2*n^2+S12*(m^4+n^4);
  SBar16 = (2*S11-2*S12-S66)*n*m^3+(2*S12-2*S22+S66)*n^3*m;
  SBar22 = S11*n^4+(2*S12+S66)*n^2*m^2+S22*m^4;
  SBar26 = (2*S11-2*S12-S66)*n^3*m+(2*S12-2*S22+S66)*n*m^3;
  SBar66 = 2*(2*S11+2*S22-4*S12-S66)*n^2*m^2+S66*(n^4+m^4);

% Assemble the SBar matrix
  SBar =[SBar11 SBar12 SBar16;
         SBar12 SBar22 SBar26;
         SBar16 SBar26 SBar66];

end
```

Stress transformation matrix $[T_\sigma]$

```
function Ts=StressTransformationMatrix(Theta)
% StressTransformationMatrix computes the 2D stress coordinate
% transformation matrix based on the Voigt notation (T_sigma)
%
%   Syntax:
%     Ts = StressTransformationMatrix(Theta)
%
%   Inputs:
```

```
%     Theta - Rotation angle in degrees
%
%   Output:
%     Ts - 3x3 2D stress coordinate transformation matrix
%          (based on the Voigt notation)
%
%   Author: Senthil S. Vel, University of Maine
%
%   See also InverseStressTransformationMatrix, StrainTransformationMatrix,
    InverseStrainTransformationMatrix

% Cosine and Sine of the angle
  m = cosd(Theta);
  n = sind(Theta);

% Stress transformation matrix T_sigma
 Ts =[ m^2  n^2   2*m*n;
        n^2  m^2  -2*m*n;
       -m*n  m*n  m^2-n^2];

end
```

Strain transformation matrix $[T_\varepsilon]$

```
function Te=StrainTransformationMatrix(Theta)
% StrainTransformationMatrix computes the 2D strain coordinate
% transformation matrix based on the Voigt notation (T_epsilon)
%
%   Syntax:
%     Te = StrainTransformationMatrix(Theta)
%
%   Inputs:
%     Theta - Rotation angle in degrees
%
%   Output:
%     Te - 3x3 2D strain coordinate transformation matrix
%          (based on the Voigt notation)
%
%   Author: Senthil S. Vel, University of Maine
%
%   See also InverseStressTransformationMatrix, StressTransformationMatrix,
    InverseStrainTransformationMatrix

% Cosine and Sine of the angle
```

```
  m = cosd(Theta);
  n = sind(Theta);

% Strain transformation matrix T_epsilon
  Te =[ m^2    n^2    m*n;
        n^2    m^2   -m*n;
      -2*m*n  2*m*n  m^2-n^2 ];

end
```

Inverse stress transformation matrix $[T_\sigma]^{-1}$

```
function [Tsinv]=InverseStressTransformationMatrix(Theta)
% InverseStressTransformationMatrix computes the inverse of the
% 2D stress coordinate transformation matrix based on the
% Voigt notation (T_sigma)
%
%   Syntax:
%     Tsinv = InverseStressTransformationMatrix(Theta)
%
%   Inputs:
%     Theta - Rotation angle in degrees
%
%   Output:
%     Tsinv - 3x3 inverse of the 2D stress coordinate transformation matrix
%         (for the Voigt notation case)
%
%   Author: Senthil S. Vel, University of Maine
%
%   See also StrainTransformationMatrix, StressTransformationMatrix,
    InverseStrainTransformationMatrix

% Cosine and Sine of the angle
  m = cosd(Theta);
  n = sind(Theta);

% Inverse of the stress transformation matrix
  Tsinv =[m^2    n^2    -2*m*n;
          n^2    m^2     2*m*n;
          m*n   -m*n    m^2-n^2];

end
```

Inverse strain transformation matrix $[T_\varepsilon]^{-1}$

```
function [Teinv]=InverseStrainTransformationMatrix(Theta)
% InverseStrainTransformationMatrix computes the inverse of the
% 2D strain coordinate transformation matrix based on the
% Voigt notation (T_epsilon)
%
%   Syntax:
%     Teinv = InverseStrainTransformationMatrix(Theta)
%
%   Inputs:
%     Theta - Rotation angle in degrees
%
%   Output:
%     Teinv - 3x3 inverse of the 2D strain coordinate transformation matrix
%         (for the Voigt notation case)
%
%   Author: Senthil S. Vel, University of Maine
%
%   See also StrainTransformationMatrix, StressTransformationMatrix,
    InverseStressTransformationMatrix

% Cosine and Sine of the angle
  m = cosd(Theta);
  n = sind(Theta);

% Inverse of the strain transformation matrix
  Teinv =[m^2      n^2     -m*n;
          n^2      m^2      m*n;
          2*m*n  -2*m*n   m^2-n^2];

end
```

Lamina engineering properties

```
function [Ex,nuxy,Ey,Gxy] = LaminaEngProperties(E1,nu12,E2,G12,Theta)
% LaminaEngProperties calculates engineering properties for an off-axis lamina
%
%   Syntax:
%     [Ex,nuxy,Ey,Gxy] = LaminaEngProperties(E1,nu12,E2,G12,Theta)
%
%   Inputs:
%     E1 - Young's modulus in the 1-direction
```

```
%     nu12 - major Poisson's ratio
%     E2 - Young's modulus in the 2-direction
%     G12 - inplane shear modulus
%     Theta - Angle in degrees from the x-axis to the 1-axis (CCW positive)
%     Note 1 and 2 are the principal material directions.
%     Typically E1, E2 and G12 are specified in SI units of Pa.
%
%   Output:
%     Ex - Young's modulus in the x-direction
%     nuxy - Poisson's ratio of an off-axis lamina
%     Ey - Young's modulus in the y-direction
%     Gxy - in-plane shear modulus of an off-axis lamina
%
%   Author: Senthil S. Vel, University of Maine
%
%   See also ReducedCompliance, ReducedStiffness, OffAxisCompliance, OffAxisStiffness

% Calculate the reduced compliance matrix S
S = ReducedCompliance(E1,nu12,E2,G12);

% Calculate the off-axis reduced compliance matrix SBar
SBar = OffAxisCompliance(S,Theta);
SBar11 = SBar(1,1);
SBar12 = SBar(1,2);
SBar22 = SBar(2,2);
SBar66 = SBar(3,3);

% Calculate the effective engineering properties of an off-axis lamina
Ex = 1/SBar11;
Ey = 1/SBar22;
nuxy = -SBar12/SBar11;
Gxy = 1/SBar66;

end
```

Tsai-Wu failure theory

```
function [Sfa, Sfr]=TsaiWu(F1t,F1c,F2t,F2c,F6,Stresses12)
% TsaiWu computes the factor of safety for a state of stress (Sfa)
% and for the reversed-in-sign state of stress (Sfr) based on the
% Tsai-Wu failure theory
%
%   Syntax:
%     [Sfa, Sfr] = TsaiWu(F1t,F1c,F2t,F2c,F6,Stresses12)
```

```
%
%   Inputs:
%     F1t - The tensile strength in the 1-material direction
%     F1c - The compressive strength in the 1-material direction
%     F2t - The tensile strength in the 2-material direction
%     F2c - The compressive strength in the 2-material direction
%     F6  - The shear strength in the 1-2 material plane
%     Stresses12 - State of stress, vector input.
%          Stresses12 = [sigma_1 sigma_2 tau_12]'.
%
%     NOTE: Consistency with the units of the strength values and the stress state
%           is required for meaningful results.
%
%   Output:
%     Sfa - factor of safety for the state of stress Stresses12
%     Sfr - factor of safety for the reversed-in-sign equivalent state of stress Stresses12
%
%   Author: Senthil S. Vel, University of Maine
%
%   See also StressTransformationMatrix, LaminaEngProperties

% Stress components
sigma1 = Stresses12(1);
sigma2 = Stresses12(2);
tau12 = Stresses12(3);

% Calculate the Tsai-Wu coefficients
f1 = 1/F1t-1/F1c;
f11 = 1/(F1t*F1c);
f2 = 1/F2t-1/F2c;
f22 = 1/(F2t*F2c);
f66 = 1/(F6^2);

% Determine the coefficients a and b
a = f11*sigma1^2+f22*sigma2^2+f66*tau12^2-sqrt(f11*f22)*sigma1*sigma2;
b = f1*sigma1+f2*sigma2;

% Determine the factor of safety for actual state of stress
Sfa=(-b+sqrt(b^2+4*a))/(2*a);

% Determine the factor of safety for reversed-in-sign state of stress
Sfr= (-b-sqrt(b^2+4*a))/(2*a);

end
```

B.3 Laminate Functions

Laminate ABD matrix

```
function [A,B,D,ABD,a,b,d,abd]=LaminateABD(N,QBarArray,ZArray)
% LaminateABD computes the [A], [B], [D] and [ABD] matrices of a laminate.
% In addition, it computes the [a], [b], [d] and [abd] matrices where [abd] is
% the inverse of [ABD].
%
%   Syntax:
%     [A,B,D,ABD,a,b,d,abd] = LaminateABD(N,QBarArray,ZArray)
%
%   Inputs:
%     N         - Number of layers in the laminate
%     QBarArray - An array where QBarArray{k} is a 3x3 matrix of off-axis
%                 stiffnesses of the kth layer of the laminate
%     ZArray    - Array of interface z-coordinates of a laminate
%
%   Output:
%     A   - [A] matrix (3x3)
%     B   - [B] matrix (3x3)
%     D   - [D] matrix (3x3)
%     ABD - [ABD] matrix (6x6)
%     a   - [a] matrix (3x3)
%     b   - [b] matrix (3x3)
%     d   - [d] matrix (3x3)
%     abd - [abd] matrix, inverse of ABD (6x6)
%
%   Author: Senthil S. Vel, University of Maine
%
%   See also LaminateStrainsXY, LaminateStressesXY, LaminateEngineeringProperties.

% Initialize the A, B and D matrices
A=zeros(3,3);
B=zeros(3,3);
D=zeros(3,3);

% Perform layer by layer summation to obtain the A, B and D matrices
for k = 1:N
   A = A + (ZArray(k+1)-ZArray(k))*QBarArray{k};
   B = B + (1/2)*((ZArray(k+1))^2-(ZArray(k))^2)*QBarArray{k};
   D = D + (1/3)*((ZArray(k+1))^3-(ZArray(k))^3)*QBarArray{k};
end
```

```
% Arrange the A, B and D into a 6x6 ABD matrix
ABD = [A B; B D];

% Find the inverse of the ABD matrix
abd = inv(ABD);
a = abd(1:3,1:3);
b = abd(1:3,4:6);
c = abd(4:6,1:3);
d = abd(4:6,4:6);

end
```

Midsurface strains and curvatures

```
function [Epsilon0,Kappa] = MidsurfaceStrainsCurvatures(abd,Nx,Ny,Nxy,Mx,My,Mxy)
% MidsurfaceStrainsCurvatures computes the mid-surface strains and curvatures
% of a lamina under loads Nx, Ny and Nxy and moments Mx, My and Mxy based on
% the load-deformation relations: [epsilon,kappa] = [abd] [N,M].
%
%   Syntax:
%     [Epsilon0,Kappa] = MidsurfaceStrainsCurvatures(abd,Nx,Ny,Nxy,Mx,My,Mxy)
%
%   Inputs:
%     abd - [abd] matrix (i.e. inverse of [ABD]), could be computed using LaminateABD
%     Nx  - x-direction axial load (force)
%     Ny  - y-direction axial load (force)
%     Nxy - xy-plane shear load (force)
%     Mx  - Bending moment about the x-axis
%     My  - Bending moment about the y-axis
%     Mxy - Twisting moment
%
%   Output:
%     Epsilon0 - Computed mid-surface strain of the laminate
%     Kappa    - Computed mid-surface curvature of the laminate
%
%   Author: Senthil S. Vel, University of Maine
%
%   See also LaminateABD, LaminateStrainsXY, LaminateStressesXY.

% Create a column array of forces and moments
NM = [Nx Ny Nxy Mx My Mxy]';

% Determine the midsurface strains and curvatures array
EpsilonKappaArray = abd*NM;
```

```
% Extract the midsurface strains and curvatures
Epsilon0 = EpsilonKappaArray(1:3,1);
Kappa = EpsilonKappaArray(4:6,1);

end
```

Determine the layer number given the z-coordinate

```
function LayerNumber = WhichLayer(N,ZArray,z)
% LayerNumber determines to which layer of the laminate a point
% with a specified z coordinate belongs
%
%   Syntax:
%     LayerNumber = WhichLayer(N,ZArray,z)
%
%   Inputs:
%     N      - Number of layers in the laminate
%     ZArray - Array of interface z coordinates
%     z      - thickness coordinate of the location for which the layer number
%              is to be determined
%
%   Output:
%     LayerNumber - The layer to which point z belongs to
%
%   Author: Senthil S. Vel, University of Maine
%
%   See also LaminateABD, LaminateStrainsXY, LaminateEngineeringProperties, LaminateStressesXY.

% Check layer by layer to see if ZArray(k) <= z <= ZArray(k+1)
for k = 1:N
    if (z >= ZArray(k)) & (z <= ZArray(k+1))
        LayerNumber = k; % assign layer number if ZCoord(k) <= z <= ZCoord(k+1);
    end
end

end
```

Laminate strains

```
function StrainsXY = LaminateStrainsXY(Epsilon0,Kappa,z)
% LaminateStrainsXY computes, for a given mid-surface strain vector
```

```
% and mid-surface curvatures vector, the strains at a specified location
% (or through-thickness coordinate) z.
%
%   Syntax:
%     StrainsXY = LaminateStrainsXY(Epsilon0,Kappa,z)
%
%   Inputs:
%     Epsilon0 - A 3x1 vector of mid-surface strains
%     Kappa    - A 3x1 vector of mid-surface curvatures
%     z        - z-coordinate of the location for calculating the x-y strains
%
%   Output:
%     StrainsXY - A 3x1 array of strains in the x-y (global) coordinate system
%
%   Author: Senthil S. Vel, University of Maine
%
%   See also LaminateABD, LaminateStressesXY, LaminateEngineeringProperties.

% Compute the strains based on the Kirchhoff assumptions
StrainsXY = Epsilon0+z*Kappa;

end
```

Laminate stresses

```
function StressesXY = LaminateStressesXY(QBarArray,ZArray,Epsilon0,Kappa,z)
% LaminateStressesXY computes, for a given mid-surface strain vector
% and mid-surface curvatures vector, the stresses at a specified location
% (or through-thickness coordinate) z.
%
%   Syntax:
%     StressesXY = LaminateStressesXY(QBarArray,ZArray,Epsilon0,Kappa,z)
%
%   Inputs:
%     QBarArray - An array where QBarArray{k} is a 3x3 matrix of off-axis
%                 stiffnesses of the kth layer of the laminate
%     ZCoord    - Array of interface locations, i.e., the beginning of each
%                 layer of the laminate
%     Epsilon0  - A 3x1 vector of mid-surface strains
%     Kappa     - A 3x1 vector of mid-surface curvatures
%     z         - z-coordinate of the location for calculating the x-y stresses
%
%   Output:
%     StressesXY - A 3x1 array of stresses in the x-y (global) coordinate system
```

```
%
%   Author: Senthil S. Vel, University of Maine
%
%   See also LaminateABD, LaminateStrainsXY, LaminateEngineeringProperties, WhichLayer.

% Determine the number of layers from the length of the QBar array
N = length(QBarArray);

% Determine which layer the z coordinate belongs to
k = WhichLayer(N,ZArray,z); % layer number

% Determine the strains at the specified z location
StrainsXY = LaminateStrainsXY(Epsilon0,Kappa,z);

% Stresses in the x-y coordinate system
StressesXY = QBarArray{k}*StrainsXY;

end
```

Plot laminate strains

```
function PlotLaminateStrains(ComponentStr,ThetaArray,ZArray,Epsilon0,Kappa,H)
% PlotLaminateStrains plots the through-the-thickness variation
% (i.e. as a function of z) of a specifiec strain component in the laminate
%
%   Syntax:
%     PlotLaminateStrains(ComponentStr,ThetaArray,ZArray,Epsilon0,Kappa,H)
%
%   Inputs:
%     ComponentStr - Strain component to be plotted, takes one of the following
%                    character array values:
%                    in the golbal/structural coordinate system: 'ex', 'ey', 'gammaxy'
%                    in the principal/material coordiante system: 'e1', 'e2', 'gamma12'
%     ThetaArray   - Nx1 vector of layer by layer fiber orientations, where N
%                    is the number of layers in the laminate.
%     ZArray       - Array of interface z-coordinates of a laminate
%     Epsilon0     - Mid-surface strains of the laminate
%     Kappa        - Mid-surface curvatures of the laminate
%     H            - Total height, i.e. thickness, of the laminate
%
%   Output:
%     No outputs for this function (except the plot outputs as figures)
%
%   Author: Senthil S. Vel, University of Maine
```

```
%
%   See also LaminateStrainsXY, PlotLaminateStresses, PlotLaminateTsaiWuSfa .

% Read global plot parameters
global LineThickness

% Determine the number of layers from the ThetaArray
N = length(ThetaArray);

% Specify the number of sampling points per layer for the plots
PointsPerLayer  = 2;

switch ComponentStr
    case 'ex'
        CoordinateSystem ='XY';
        Component = 1;
        xstr = 'Normal strain $\varepsilon_{\textit{x}}$ ($\mu \varepsilon$)';
        FigName = 'Normal strain e_x';
    case 'ey'
        CoordinateSystem ='XY';
        Component = 2;
        xstr = 'Normal strain $\varepsilon_{\textit{y}}$ ($\mu \varepsilon$)';
        FigName = 'Normal strain e_y';
    case 'gammaxy'
        CoordinateSystem ='XY';
        Component = 3;
        xstr = 'Shear strain $\gamma_{\textit{xy}}$ ($\mu$rad)';
        FigName = 'Shear strain gamma_xy';
    case 'e1'
        CoordinateSystem ='12';
        Component = 1;
        xstr = 'Normal strain $\varepsilon_{1}$ ($\mu \varepsilon$)';
        FigName = 'Normal strain e_1';
    case 'e2'
        CoordinateSystem ='12';
        Component = 2;
        xstr = 'Normal strain $\varepsilon_{2}$ ($\mu \varepsilon$)';
        FigName = 'Normal strain e_2';
    case 'gamma12'
        CoordinateSystem ='12';
        Component = 3;
        xstr = 'Shear strain $\gamma_{12}$ ($\mu$rad)';
        FigName = 'Shear strain gamma_12';
end
```

```
% Prepare the figure
figure('Name',FigName);
clf;
hold on;

for k = 1:N
    zloc = linspace(ZArray(k),ZArray(k+1),PointsPerLayer);

    % Evaluate the strains at the sampling points
    for n = 1: PointsPerLayer
        switch CoordinateSystem
            case 'XY'
                % Evaluate strains in the XY coordinate system
                StrainColumnArray = LaminateStrainsXY(Epsilon0,Kappa,zloc(n));
            case '12'
                % Evaluate strains in the XY coordinate system
                StrainsXY = LaminateStrainsXY(Epsilon0,Kappa,zloc(n));
                % Transform strains to the 1-2 coordinate system
                Te=StrainTransformationMatrix(ThetaArray(k));
                StrainColumnArray = Te*StrainsXY;
        end
        % Extract the strain component of interest
        Strain(n) = StrainColumnArray(Component);
    end

    % plot the strain variation in layer k
    plot(Strain/1e-6,zloc/H,'k-','LineWidth',LineThickness,'LineJoin','round');

end

% Insert axes labels
xlabel(xstr,'Interpreter','latex');
ylabel('$z/H$','Interpreter','latex');

% Format the laminate plot
FormatLaminatePlot(ZArray)

end
```

Plot laminate stresses

```
function PlotLaminateStresses(ComponentStr,ThetaArray,QBarArray,ZArray,Epsilon0,Kappa,H)
% PlotLaminateStressess plots the through-the-thickness variation
% (i.e. as a function of z) of a specifiec stress component in the laminate
```

```
%
%   Syntax:
%     PlotLaminateStresses(ComponentStr,ThetaArray,QBarArray,ZArray,Epsilon0,Kappa,H)
%
%   Inputs:
%     ComponentStr - Stress component to be plotted, takes one of the following
%                    character array values:
%                    in the golbal/structural coordinate system: 'sx', 'sy', 'tauxy'
%                    in the principal/material coordiante system: 's1', 's2', 'tau12'
%     ThetaArray   - Nx1 vector of layer fiber orientations, where N is the number
%                    of layers in the laminate.
%     QBarArray    - An array where QBarArray{k} is a 3x3 matrix of off-axis
%                    stiffnesses of the kth layer of the laminate
%     ZArray       - Array of interface z-coordinates of a laminate
%     Epsilon0     - Mid-surface strains of the laminate
%     Kappa        - Mid-surface curvatures of the laminate
%     H            - Total height, i.e. thickness, of the laminate
%
%   Output:
%     No outputs for this function (except the plot outputs as figures)
%
%   Author: Senthil S. Vel, University of Maine
%
%   See also LaminateStressesXY, PlotLaminateStrains, PlotLaminateTsaiWuSfr .

% Read global plot parameters
global LineThickness

% Determine the number of layers from the ZArray
N = length(QBarArray);

% Specify the number of sampling points per layer for the plots
PointsPerLayer  = 2;

switch ComponentStr
    case 'sx'
        CoordinateSystem ='XY';
        Component = 1;
        xstr = 'Normal stress $\sigma_{\textit{x}}$ (MPa)';
        FigName = 'Normal stress sigma_x';
    case 'sy'
        CoordinateSystem ='XY';
        Component = 2;
        xstr = 'Normal stress $\sigma_{\textit{y}}$ (MPa)';
        FigName = 'Normal stress sigma_y';
```

```
    case 'tauxy'
        CoordinateSystem ='XY';
        Component = 3;
        xstr = 'Shear stress $\tau_{\textit{xy}}$ (MPa)';
        FigName = 'Shear stress tau_xy';
    case 's1'
        CoordinateSystem ='12';
        Component = 1;
        xstr = 'Normal stress $\sigma_{1}$ (MPa)';
        FigName = 'Normal stress sigma_1';
    case 's2'
        CoordinateSystem ='12';
        Component = 2;
        xstr = 'Normal stress $\sigma_{2}$ (MPa)';
        FigName = 'Normal stress sigma_2';
    case 'tau12'
        CoordinateSystem ='12';
        Component = 3;
        xstr = 'Shear stress $\tau_{12}$ (MPa)';
        FigName = 'Shear stress tau_12';
end

% Clear figure and hold while plotting
figure('Name',FigName);
clf;
hold on;

for k = 1:N
    % Sample points from Z(k)-eps to Z(k+1)+eps. The eps is used to
    % avoid ambiguity of which layer the interface belongs to. Use a very
    % small value for the parameter eps to ensure points close to the
    % interfaces are included in the plots
    zloc = linspace(ZArray(k)+eps,ZArray(k+1)-eps,PointsPerLayer);

    % Evaluate the stresses at the sampling points
    for n = 1: PointsPerLayer
        switch CoordinateSystem
            case 'XY'
              % Evaluate stresses in the XY coordinate system
              StressColumnArray = LaminateStressesXY(QBarArray,ZArray,Epsilon0,Kappa,zloc(n));
            case '12'
              % Evaluate stresses in the XY coordinate system
              StressesXY = LaminateStressesXY(QBarArray,ZArray,Epsilon0,Kappa,zloc(n));
              % Transform stresses to the 1-2 coordinate system
              Ts=StressTransformationMatrix(ThetaArray(k));
```

```
            StressColumnArray = Ts*StressesXY;
        end
        % Extract the stress component of interest
        Stress(n) = StressColumnArray(Component);
    end

    % plot the stress variation in layer k
    hp = plot(Stress/1e6,zloc/H,'k-','LineWidth',LineThickness,'LineJoin','round');
end

% Insert axes labels
xlabel(xstr,'Interpreter','latex');
ylabel('$z/H$','Interpreter','latex');

% Format the laminate plot
FormatLaminatePlot(ZArray)

end
```

Plot Tsai-Wu safety factor S_{fa}

```
function [SfaMin,kmin,zmin] = PlotLaminateTsaiWuSfa(F1t,F1c,F2t,F2c,F6,ThetaArray,QBarArray,
    ZArray,Epsilon0,Kappa,H)
% PlotLaminateTsaiWuSfa plots the through-the-thickness variation (i.e. as a function of z)
% of the Tsai-Wu factor of safety for given mid-surface strains and curvatures in a laminate.
% It also outputs the minimum factor of safety, its z-location and the layer of
% the laminate it occurs in.
%
%   Syntax:
%     PlotLaminateTsaiWuSfa(F1t,F1c,F2t,F2c,F6,ThetaArray,QBarArray,ZArray,Epsilon0,Kappa,H)
%
%   Inputs:
%     F1t        - Tensile strength in the 1-direction (principal coordinate system)
%     F1c        - Compressive strength in the 1-direction (principal coordinate system)
%     F2t        - Tensile strength in the 2-direction (principal coordinate system)
%     F2c        - Compressive strength in the 2-direction (principal coordinate system)
%     F6         - Shear strength in the 1-2 plane (principal coordinate system)
%     ThetaArray - Nx1 vector of layer fiber orientations, where N is the number
%                  of layers in the laminate.
%     QBarArray  - An array where QBarArray{k} is a 3x3 matrix of off-axis
%                  stiffnesses of the kth layer of the laminate
%     ZArray     - Array of interface z-coordinates of a laminate
%     Epsilon0   - Mid-surface strains of the laminate
%     Kappa      - Mid-surface curvatures of the laminate
```

```
%     H          - Total height, i.e. thickness, of the laminate
%
%   Output:
%     SfaMin - Minimum Tsai-Wu factor of safety
%     kmin   - Minimum Tsai-Wu factor of safety layer of occurence
%     zmin   - Minimum Tsai-Wu factor of safety location (height) of occurence
%     Also the plots are an output
%
%   Author: Senthil S. Vel, University of Maine
%
%   See also PlotLaminateStresses, PlotLaminateStrains, PlotLaminateTsaiWuSfr .

% Read global plot parameters
global LineThickness

% Determine the number of layers from the QBarArray
N = length(QBarArray);

% Specify the number of sampling points per layer for the plots
PointsPerLayer  = 500;

% Prepare the figure
figure('Name','Safety factor Sfa');
clf;
hold on;

% Initialize the minimum safety factor value
SfaMin = inf;

for k = 1:N
    % Sample points from Z(k)-eps to Z(k+1)+eps. The eps is used to
    % avoid ambiguity of which layer the interface belongs to. Use a very
    % small value for the parameter eps to ensure points close to the
    % interfaces are included in the plots
    eps = 1e-12;
    zloc = linspace(ZArray(k)+eps,ZArray(k+1)-eps,PointsPerLayer);

    % Evaluate the safety factor at the sampling points
    for n = 1: PointsPerLayer
        % Evaluate stresses in the XY coordinate system
        StressesXY = LaminateStressesXY(QBarArray,ZArray,Epsilon0,Kappa,zloc(n));

        % Transform stresses to the 1-2 coordinate system
        Ts=StressTransformationMatrix(ThetaArray(k));
        Stresses12 = Ts*StressesXY;
```

```
        % Calculate the safety factor
        [Sfa(n), Sfr(n)]=TsaiWu(F1t,F1c,F2t,F2c,F6,Stresses12);

        % Update the min safety factor values
        if Sfa(n) < SfaMin
            SfaMin = Sfa(n);
            zmin = zloc(n);
            kmin = k;
        end

        %SfaMin = min(SfaMin, min(Sfa));
    end

    % plot the safety factor variation in layer k
    hp = plot(Sfa,zloc/H,'k-','LineWidth',LineThickness);
end

% Insert axes labels
xlabel('Safety factor $S_{fa}$','Interpreter','latex');
ylabel('$z/H$','Interpreter','latex');

% Format the laminate plot
FormatLaminatePlot(ZArray);

end
```

Plot Tsai-Wu safety factor S_{fr}

```
function [SfrMin,kmin,zmin] = PlotLaminateTsaiWuSfr(F1t,F1c,F2t,F2c,F6,ThetaArray,QBarArray,
    ZArray,Epsilon0,Kappa,H)
% PlotLaminateTsaiWuSfr plots the through-the-thickness variation (i.e. as a function of z)
% of the Tsai-Wu reversed-in-sign factor of safety for given mid-surface strains and
% curvatures in a laminate. It also outputs the minimum factor of safety, its
% z-location and the layer of the laminate it occurs in.
%
%   Syntax:
%     PlotLaminateTsaiWuSfr(F1t,F1c,F2t,F2c,F6,ThetaArray,QBarArray,ZArray,Epsilon0,Kappa,H)
%
%   Inputs:
%     F1t        - Tensile strength in the 1-direction (principal coordinate system)
%     F1c        - Compressive strength in the 1-direction (principal coordinate system)
%     F2t        - Tensile strength in the 2-direction (principal coordinate system)
%     F2c        - Compressive strength in the 2-direction (principal coordinate system)
```

```
%     F6         - Shear strength in the 1-2 plane (principal coordinate system)
%     ThetaArray - Nx1 vector of layer fiber orientations, where N is the number
%                  of layers in the laminate.
%     QBarArray  - An array where QBarArray{k} is a 3x3 matrix of off-axis
%                  stiffnesses of the kth layer of the laminate
%     ZArray     - Array of interface z-coordinates of a laminate
%     Epsilon0   - Mid-surface strains of the laminate
%     Kappa      - Mid-surface curvatures of the laminate
%     H          - Total height, i.e. thickness, of the laminate
%
%   Output:
%     SfrMin - Minimum Tsai-Wu factor of safety (reversed-in-sign)
%     kmin   - Minimum Tsai-Wu factor of safety (reversed-in-sign) layer of occurence
%     zmin   - Minimum Tsai-Wu factor of safety (reversed-in-sign) location (height) of
    occurence
%     Also the plots are an output
%
%   Author: Senthil S. Vel, University of Maine
%
%   See also PlotLaminateStresses, PlotLaminateStrains, PlotLaminateTsaiWuSfa .

% Read global plot parameters
global LineThickness

% Determine the number of layers from the QBarArray
N = length(QBarArray);

% Specify the number of sampling points per layer for the plots
PointsPerLayer  = 500;

% Prepare the figure
figure('Name','Reversed-in-sign safety factor |Sfr|');
clf;
hold on;

% Initialize the minimum safety factor value
SfrMin = inf;

for k = 1:N
    % Sample points from Z(k)-eps to Z(k+1)+eps. The eps is used to
    % avoid ambiguity of which layer the interface belongs to. Use a very
    % small value for the parameter eps to ensure points close to the
    % interfaces are included in the plots
    eps = 1e-12;
    zloc = linspace(ZArray(k)+eps,ZArray(k+1)-eps,PointsPerLayer);
```

```
    % Evaluate the safety factor at the sampling points
    for n = 1: PointsPerLayer
        % Evaluate stresses in the XY coordinate system
        StressesXY = LaminateStressesXY(QBarArray,ZArray,Epsilon0,Kappa,zloc(n));

        % Transform stresses to the 1-2 coordinate system
        Ts=StressTransformationMatrix(ThetaArray(k));
        Stresses12 = Ts*StressesXY;

        % Calculate the safety factor
        [Sfa(n), Sfr(n)]=TsaiWu(F1t,F1c,F2t,F2c,F6,Stresses12);

        % Update the min safety factor values
        if abs(Sfr(n)) < SfrMin
            SfrMin = abs(Sfr(n));
            zmin = zloc(n);
            kmin = k;
        end
    end

    % plot the safety factor variation in layer k
    plot(abs(Sfr),zloc/H,'k-','LineWidth',LineThickness);
end

% Insert axes labels
xlabel('Safety factor $|S_{fr}|$','Interpreter','latex');
ylabel('$z/H$','Interpreter','latex');

% Format the laminate plot
FormatLaminatePlot(ZArray);

end
```

Format laminate through-thickness plots

```
function FormatLaminatePlot(ZArray)
% FormatLaminatePlot sets up the figure characteristics
% for the through-the-thickness plots for a laminate
%
%   Syntax:
%     FormatLaminatePlot(ZArray, FontSize)
%
%   Inputs:
```

```
%     ZArray - Array of interface z-coordinates of a laminate
%
%   Output:
%     No output, the plot through-thickness aesthetics will be modified
%
%   Author: Senthil S. Vel, University of Maine

% Read global variables
global LineThickness FontSize
global InterfaceLineColor MidsurfaceLineColor VerticalAxisLineColor

% Number of layers
N = length(ZArray)-1;

% Laminate thickness
H = ZArray(N+1)-ZArray(1);

% Set the plot box aspect ratio to the golden ratio
pbaspect([1.618 1 1])

% Get axis range
V = axis;

% Change the axis settings to make the figure more readable
ha = gca;
set(ha,'Box','on');
set(ha,'FontSize',FontSize);
set(ha,'LineWidth',0.7*LineThickness);

% Draw the mid-surface
x =[V(1); V(2)];  % The line extends over the entire horizontal range
y = [0;0];
hl = line(x,y);
set(hl,'LineStyle','--')
set(hl,'Color',MidsurfaceLineColor);
set(hl,'LineWidth',LineThickness/4);
uistack(hl,'bottom');

% Plot horizontal lines corresponding to the bottom surface, top
% surface and interface
for k = 2:N
    x =[V(1); V(2)];  % The line extends over the entire horizontal range
    y = [ZArray(k)/H; ZArray(k)/H];
    hl = line(x,y);
    set(hl,'Color',InterfaceLineColor);
```

```
    set(hl,'LineWidth',LineThickness/4);
    uistack(hl,'bottom');
    %get(hl)
    %set(hl,'Visible')
end

% Draw the vertical line corresponding to VAxisValue
x =[0; 0];
y = [V(3);V(4)]; % The line extends over the entire vertical range
hl = line(x,y);
set(hl,'LineStyle','-')
set(hl,'Color',VerticalAxisLineColor);
set(hl,'LineWidth',LineThickness/4);
uistack(hl,'bottom');

% Show labels for the interface locations if there are less than 10 layers
if N < 10
    % Set yticks
    yticks(ZArray/H);

    % Set ytick labels
    [Num,Den]=rat(ZArray/H);
    for k = 1:N+1
        if Num(k) == 0
            YTickLabelStr{k} = '0';
        else
            YTickLabelStr{k} =strcat(num2str(Num(k)),'/',num2str(Den(k)));
        end
        yticklabels(YTickLabelStr)
    end
else
    % Show only the bottom surface, midsurface and top surface labels
    % if there are more than 10 layers so that the ytick labels are not
    % too cluttered.
    yticks([-1/2 0 1/2]);
end

% Reset axis range
axis(V);
end
```

B.4 Sample Analysis Scripts

Lamina analysis

```
%***************************************************************************
%                   * Sample lamina analysis script *
%
%          Calculates the response of an off-axis lamina to
%         prescribed stresses in the global coordinate system
%***************************************************************************

%% Clear variables and close all figures
clearvars
close all

%% Read lamina properties
[E1,nu12,E2,G12,F1t,F1c,F2t,F2c,F6,h,rho] = UnidirectionalCarbonEpoxyProperties;

%% Specify the ply orientation in degrees
Theta = 30;
fprintf('Lamina orientation theta = %g degrees \n\n',Theta)

%% Specify the stresses in the X-Y coordinate system
StressesXY = [225; 50; 50]*1e6;
disp('Stresses in the X-Y coordinate system (MPa):')
disp(StressesXY/1e6)

%% Calculate the reduced compliance matrix
S = ReducedCompliance(E1,nu12,E2,G12);
disp('Reduced compliance S (TPa^-1)=');  disp([S]*1e12)

%% Calculate the reduced stiffness matrix
Q = ReducedStiffness(E1,nu12,E2,G12);
disp('Reduced stiffness Q (GPa) =');  disp([Q]/1e9)

%% Compute the off-axis reduced compliance matrix
SBar = OffAxisCompliance(S,Theta);
disp('Off-axis compliance SBar (TPa^-1)=')
disp([SBar]*1e12)

%% Compute the off-axis reduced stiffness matrix
QBar = OffAxisStiffness(Q,Theta);
disp('Off-axis stiffness QBar (GPa) =')
disp([QBar]/1e9)
```

```
%% Compute the stress transformation matrix
Ts=StressTransformationMatrix(Theta);
disp('Stress transformation matrix Ts:')
disp(Ts)

%% Compute the stresses in the 1-2 coordinate system
Stresses12 = Ts*StressesXY;
disp('Stresses in the 1-2 coordinate system (MPa):')
disp(Stresses12/1e6)

%% Calculate the strains in the X-Y coordinate system
StrainsXY = SBar*StressesXY;
disp('Strains in the X-Y coordinate system (micro m/m):')
disp(StrainsXY/1e-6)

%% Compute the strain transformation matrix
Te=StrainTransformationMatrix(Theta);
disp('Strain transformation matrix Te:')
disp(Te)

%% Calculate the strains in the 1-2 coordinate system
Strains12 = Te*StrainsXY;
disp('Strains in the 1-2 coordinate system (micro m/m):')
disp(Strains12/1e-6)

%% Calculate the safety factor using the Tsai-Wu failure theory
[Sfa, Sfr]=TsaiWu(F1t,F1c,F2t,F2c,F6,Stresses12);
fprintf('Tsai-Wu safety factor for actual stress state: Sfa = %g \n\n',Sfa)
fprintf('Tsai-Wu safety factor for reversed-in-sign stress state: Sfr = %g \n\n',Sfr)
```

Laminate analysis

```
%***********************************************************************
%                 * Sample laminate analysis script *
%
%        Calculates the response of a representative laminate to
%                 prescribed force and moment resultants
%***********************************************************************

%% Clear variables and close all figures
clearvars
close all
```

```
%% Read lamina properties
[E1,nu12,E2,G12,F1t,F1c,F2t,F2c,F6,h,rho] = UnidirectionalCarbonEpoxyProperties;

%% Specify the stacking sequence of the plies
ThetaArray = [0 -45 90];
fprintf('Laminate stacking sequence Theta = %s (degrees) \n\n',strcat('[',num2str(ThetaArray),'
    ]'))

%% Specify the force resultants per unit width (N/m)
Nx = 0;
Ny = 0;
Nxy = 0;

%% Specify the momment resultants per unit width (N.m/m)
Mx = 1.0;
My = 0;
Mxy = 0;

%% Determine the number of layers
N = length(ThetaArray);
fprintf('Number of layers: %g \n\n',N)

%% Compute total laminate thickness H
H = N*h;
fprintf('Laminate thickness H = %g mm \n\n',H/1e-3)

%% Evaluate laminate interface locations Z_k
for k = 1:N+1
    ZArray(k)=-H/2+(k-1)*h;
end
%fprintf('Laminate interface locations Z = %s mm \n\n',strcat('[',num2str(ZCoord/1e-3),']'))
disp('Laminate interface locations Z (mm) =');
disp(strcat('[',num2str(ZArray/1e-3),']'));disp(' ')

%% Calculate the reduced compliance matrix
S = ReducedCompliance(E1,nu12,E2,G12);
disp('Reduced compliance matrix S (TPa^-1)=');
disp([S]*1e12)

%% Calculate the reduced stiffness matrix
Q = ReducedStiffness(E1,nu12,E2,G12);
disp('Reduced stiffness matrix Q (GPa) =');
disp([Q]/1e9)

%% Compute the off-axis reduced stiffness matrices
```

```
for k = 1:N
 QBarArray{k}=OffAxisStiffness(Q,ThetaArray(k));
 disp(strcat('Off-axis stiffness QBar{',num2str(k),'} (GPa) ='));
 disp(QBarArray{k}/1e9)
end

%% Compute laminate ABD stiffness matrix
[A,B,D,ABD,a,b,d,abd]= LaminateABD(N,QBarArray,ZArray);
disp('A (10^6 N/m):'),  disp(A/1e6)
disp('B (N):'),  disp(B)
disp('D (10^-3 N-m):'), disp(D/1e-3)
disp('a (10^-9 m/N):'), disp(a/1e-9)
disp('b (10^-3 1/N):'), disp(b/1e-3)
disp('d (1/N-m):'),     disp(d)

%% Compute the midsurface strains and curvatures
[Epsilon0,Kappa] = MidsurfaceStrainsCurvatures(abd,Nx,Ny,Nxy,Mx,My,Mxy);
disp('Midsurface strains Epsilon0 (micro):')
disp(Epsilon0/1e-6)
disp('Midsurface curvatures Kappa (1/m):')
disp(Kappa)

%% Calculate the strains and stresses at the z location of interest
z = H/4;  % Sample z-location
fprintf('z coordinate of interest = %g mm \n\n',z/1e-3)

k = WhichLayer(N,ZArray,z);
fprintf('z coordinate belongs to layer %g \n\n',k)

StrainsXY = LaminateStrainsXY(Epsilon0,Kappa,z);
disp('StrainsXY at the z location (micro):')
disp(StrainsXY/1e-6)

StressesXY = LaminateStressesXY(QBarArray,ZArray,Epsilon0,Kappa,z);
disp('StressesXY at the z location (MPa):')
disp(StressesXY/1e6)

%% Plot the through-thickness variation of strains
PlotLaminateStrains('ex',ThetaArray,ZArray,Epsilon0,Kappa,H)

PlotLaminateStrains('ey',ThetaArray,ZArray,Epsilon0,Kappa,H)

PlotLaminateStrains('gammaxy',ThetaArray,ZArray,Epsilon0,Kappa,H)

PlotLaminateStrains('e1',ThetaArray,ZArray,Epsilon0,Kappa,H)
```

```
PlotLaminateStrains('e2',ThetaArray,ZArray,Epsilon0,Kappa,H)

PlotLaminateStrains('gamma12',ThetaArray,ZArray,Epsilon0,Kappa,H)

%% Plot the through-thickness variation of stresses
PlotLaminateStresses('sx',ThetaArray,QBarArray,ZArray,Epsilon0,Kappa,H)

PlotLaminateStresses('sy',ThetaArray,QBarArray,ZArray,Epsilon0,Kappa,H)

PlotLaminateStresses('tauxy',ThetaArray,QBarArray,ZArray,Epsilon0,Kappa,H)

PlotLaminateStresses('s1',ThetaArray,QBarArray,ZArray,Epsilon0,Kappa,H)

PlotLaminateStresses('s2',ThetaArray,QBarArray,ZArray,Epsilon0,Kappa,H)

PlotLaminateStresses('tau12',ThetaArray,QBarArray,ZArray,Epsilon0,Kappa,H)

%% Plot the through-thickness variation of the safety factor Sfa
[SfaMin,kmin,zmin] = PlotLaminateTsaiWuSfa(F1t,F1c,F2t,F2c,F6,ThetaArray,QBarArray,ZArray,
    Epsilon0,Kappa,H);
fprintf('Minimum laminate safety factor: SfaMin = %g \n',SfaMin)
fprintf('Layer where SfaMin occurs: k = %g \n',kmin)
fprintf('Location where SfaMin occurs: z = %g mm \n\n',zmin/1e-3)

%% Plot the through-thickness variation of the safety factor Sfr
[SfrMin,kmin,zmin] = PlotLaminateTsaiWuSfr(F1t,F1c,F2t,F2c,F6,ThetaArray,QBarArray,ZArray,
    Epsilon0,Kappa,H);
fprintf('Minimum laminate reversed-in-sign safety factor: |SfrMin| = %g \n',SfrMin)
fprintf('Layer where SfrMin occurs: k = %g \n',kmin)
fprintf('Location where SfrMin occurs: z = %g mm \n\n',zmin/1e-3)
```

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